chapter 8

Shape decompositions for visual recognition: the role of transversality

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1. INTRODUCTION

Most theories of shape recognition agree that to recognize a complex shape it is useful to decompose the shape into simpler parts. The reasons are straightforward (Hoffman & Richards 1984). One never sees all of an opaque object at once; certainly its back is not visible, and even its front may be partially occluded by objects interposed between it and the viewer. So unless one can afford the luxury of seeing an object in its entirety before recognizing it (perhaps by walking around it), one must recognize objects from only partial information. In addition, some objects have moveable parts, such as arms or fingers, which allow them to assume many configurations. Decomposing such objects into appropriate parts, thereby decoupling configuration from other aspects of their shapes, can make easier their recognition. Finally, a

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classification or description of parts, if the parts are appropriately chosen, is likely to be simpler than a classification of arbitrary shapes, and indeed should contribute to such a more general classification.

Although most theories agree that parts, in principle, are useful for recognition, they often disagree about how parts should be defined. This despite the widely acknowledged constraints that the parts on an object, however the parts are defined, should not in general change with minor changes in viewing geometry, i.e., with minor changes in the relative positions of the object and viewer, nor should the parts change with minor changes in overall size of the object.

There are two distinct approaches to the problem of part definition. The first, and by far most common, defines parts by their shapes; the second defines parts by their boundaries. The most typical parts proposed by partisans of the first approach, henceforth the primitive-based approach, are cylinders, cones, spheres, and polyhedra. Cylinders and cones are quite useful parts for representing the shapes of animals since many of their limbs are roughly cylindrical; polyhedra, on the other hand, do quite nicely for many buildings, books, and some furniture. Once the primitive part shapes are stipulated, it then remains to determine how to find these parts in complex objects, how to represent the more metrical aspects of the primitive parts (e.g., their length and width), and how to assign predicates of spatial relationships among parts (e.g., above, inside, to the right of).

To date there are but two theories representative of the second, or boundary-based, approach. Koenderink and van Doorn (1980, 1982) were the first to suggest that parts should be defined by their boundaries (though they were studying shading, not shape recognition); they propose that the appropriate boundaries are parabolic contours, i.e., contours on a surface where the Gaussian curvature is zero. Such contours possess several attractive properties. For instance, parabolic lines do not intersect and are always closed contours. By choosing them as part boundaries, Koenderink and van Doorn find that, on smooth surfaces of genus zero (no holes), there are only four qualitative classes of parts, which they call humps, dimples, furrows, and ridges.

Hoffman and Richards (1984) also suggest that parts should be defined by their boundaries rather than by a prespecified set of primitive shapes; they propose that part boundaries, instead of being defined by parabolic curves, should be defined by contours of negative minima of principal curvatures along lines of curvature, or in some cases, by contours of positive maxima of principal curvatures along lines of curvature. Hoffman and Richards’ proposal will be discussed and extended further in this paper.

What are the relative merits of the boundary-based and primitive-based approaches to the problem of part definition? If one wants only to recognize a limited class of objects, say animals or aircraft, then the primitive-based approach is quite satisfactory. If, on the other hand, one wants a general pur-

pose shape recognizer, then the primitive-based approach is the better choice. The simplest reason that most shapes—human heads, cones, spheres, pol—do not look like a single part, as exemplified by van Doorn, does give a part definition which is guaranteed to exist on any smooth surface (with some behavior at the surface. Once defined, it becomes a differential geometric part of parts that can indeed arise. Although the boundary-based approach has much to recommend it, all of the rules and one must be careful, if one’s partitioning rule is not primarily for relevance to the recognition task. (Koenderink’s analysis of shading, not shape recognition, and the role of transversality (in the) definition of part perception of parts which follow the spuriousness rule is easily disconfirmed. Figures 1 and 2). According to the par theorems, surfaces may change on a surface when the parabolic lines remains unchanged. (2) Surfaces since these surfaces have zero curvature on a cosine surface, surface discontinuity is a ruinous circular contours in the figure initially successive ring-like parts. Now turn the dotted circular contours no longer lie on top of them. In effect, the locus change in figure and ground induced by a cylindrical surface, which has zero G a surface should have no parts (or infinite hypothesis because every point on the is. However we do see a small number of
pose shape recognizer, then the primitive-based approach is inadequate for the simple reason that most shapes—human faces for instance—are not composed merely of cylinders, cones, spheres, polyhedra, or some combination of these. And adding new primitives as needed to handle new objects one encounters is hardly a way to build a principled theory. However, the boundary-based approach, as exemplified by the parabolic lines rule of Koenderink and van Doorn, does give a part definition which is completely general, for such lines are guaranteed to exist on any smooth compact surface (ovoids being the single, and easily handled, exception) and to provide a complete and well-behaved partition of the surface. Once the boundaries of parts have been so specified, it becomes a differential geometrical investigation to determine the kinds of parts that can indeed arise. This Koenderink and van Doorn have already done for compact smooth surfaces of genus zero, and there is no reason the investigation need be restricted from more complicated classes of surfaces. In this way a completely general, and principled, theory of part definition and part description can be obtained. In a sense, the primitive-based approach confuses the problem of part definition with the separate problem of part description, taking the latter to be the former.

Although the boundary-based approach taken by Koenderink and van Doorn has much to recommend it, there are many possible boundary-based rules and one must be careful, if one’s goal is shape recognition, to choose a partitioning rule not primarily for mathematical convenience but for its relevance to the recognition task. (Koenderink & van Doorn’s goal was an analysis of shading, not shape recognition.) In this regard, we will discuss shortly the role of transversality in constructing a definition of part boundary. Here we simply note that the human visual system does not appear to employ parabolic lines in its definition of parts, because two predictions about our perception of parts which follow straightforwardly from a parabolic lines partitioning rule are easily disconfirmed (the two exceptions are illustrated in Figures 1 and 2). According to the parabolic contours rule (1) part locations should not change on a surface when figure and ground reverse since the parabolic lines remains unchanged and (2) no parts should be seen on cylindrical surfaces since these surfaces have zero Gaussian curvature everywhere. Figure 1, a cosine surface, disconfirms the first prediction. Notice that the dotted circular contours in the figure initially appear to lie in the troughs between successive ring-like parts. Now turn the figure upside-down and note that the dotted circular contours no longer lie between the ring-shaped parts but rather lie on top of them. In effect, the location of the parts has changed with a change in figure and ground induced by inverting the surface. Figure 2 depicts a cylindrical surface, which has zero Gaussian curvature at every point. Such a surface should have no parts (or infinitely many parts) by the parabolic lines hypothesis because every point of the surface should lie on a part boundary. However we do see a small number of parts in this shape, which disconfirms
the second prediction. N hump-like parts with part lines are drawn. Again, if zation into parts can be se within parts.

2. MOTIVATION OF

The motivation for the \( \gamma \) Hoffman and Richards (1) and arbitrarily shaped obj Figure 3. Surely these tw allow one of the objects thus forming a new comp are good candidates for p of points where the surf is a good candidate for the

Is there any special prc be used to identify the lc and thereby to identify the two surfaces intersect the means that the tangent pla orientations at each point that there is a discontinu composite object at each 3). If the two original surf when the discontinuity is which is the case illustrate original surface is remov depression bounded by a as is illustrated in Figure 3.

This intuitive descrip Consider two compact ob

Figure 3. Transversal inte
the second prediction. Notice that this surface appears to be composed of hump-like parts with part boundaries located approximately where the dotted lines are drawn. Again, if the figure is turned upside-down a different organization into parts can be seen. The dotted lines no longer lie between parts but within parts.

2. MOTIVATION OF PARTITIONING RULES

The motivation for the partitioning rule proposed by Hoffman (1983) and Hoffman and Richards (1984) is roughly as follows. Consider two separate and arbitrarily shaped objects in a visual scene, as shown in the left half of Figure 3. Surely these two 3-D objects are separate parts of the scene. Now allow one of the objects to penetrate the other at some arbitrary orientation, thus forming a new composite object. Then certainly the two original objects are good candidates for parts of the resulting composite object, and the locus of points where the surface of the first object meets the surface of the second is a good candidate for the part boundary.

Is there any special property about the way two surfaces intersect that can be used to identify the locus of their intersection on the composite surface, and thereby to identify the boundary between the parts? Indeed there is: when two surfaces intersect they intersect transversally with probability one. This means that the tangent planes to the two intersecting surfaces are of different orientations at each point where the surfaces intersect. This implies further that there is a discontinuity of the tangent plane to the surface of the new composite object at each point along the contour of intersection (see Figure 3). If the two original surfaces are left together to form the composite surface when the discontinuity is concave at each point on the contour of intersection, which is the case illustrated in Figure 3. If, on the other hand, one of the two original surface is removed subsequent to penetrating the other, it leaves a depression bounded by a contour of convex discontinuity of the tangent plane, as is illustrated in Figure 4.

This intuitive description can be made more precise in the following way. Consider two compact objects, say obj₁ and obj₂, whose surfaces are given,

![Figure 3. Transversal intersection leading to a protruding part](image-url)
Figure 4. Transversal intersection leading to a protruding part

respectively, as the zero level sets of the two functions \( f_1(x) \) and \( f_2(x) \). Here \( x = [x, y, z] \in \mathbb{R}^3 \). (A level set of a function, \( f \), corresponding to some constant, \( c \), is the set of all points, \( Q \) where \( f(Q) = c \). The functions \( f_1 \) and \( f_2 \) are sometimes called “inside-outside” functions in the computer graphics literature (Barr 1983; Blinn 1982), because they can be used to define which points in \( \mathbb{R}^3 \) are inside the corresponding object and which are outside:

\[
\text{obj}_1 = \{ x \in \mathbb{R}^3 | f_1(x) \leq 0 \}
\]
\[
\text{obj}_2 = \{ x \in \mathbb{R}^3 | f_2(x) \leq 0 \}
\]

That is, points in \( \mathbb{R}^3 \) for which the inside-outside function is negative or zero constitute the object, whereas points for which the function is positive are outside. For instance, \( \text{obj}_1 \) might be a sphere defined by the function

\[
f_1(x) = x^2 + y^2 + z^2 - 1
\]

Points for which \( f_1 \) is negative lie inside the sphere, points for which \( f_1 \) is zero constitute its surface, and points where \( f_1 \) is positive lie outside.

The new composite object formed by interpenetrating \( \text{obj}_1 \) with \( \text{obj}_2 \) and leaving the two together can be defined as the closed-set solid union of \( \text{obj}_1 \) and \( \text{obj}_2 \):

\[
\text{obj}_{\text{new}} = \text{obj}_1 \cup \text{obj}_2 = \{ x \in \mathbb{R}^3 | x \in \text{obj}_1 \text{ OR } x \in \text{obj}_2 \}
\]
\[
= \{ x \in \mathbb{R}^3 | f_1(x) \leq 0 \text{ OR } f_2(x) \leq 0 \}.
\]

The surface of the new composite object is then (Barr, 1983)

\[
Surf(\text{obj}_1 \cup \text{obj}_2) = \{ x \in \mathbb{R}^3 | x \in Surf(\text{obj}_1) \& x \in \text{obj}_2 \text{ OR } x \notin \text{obj}_1 \& x \in Surf(\text{obj}_2) \}
\]

This new composite surface, consisting of points \( v \) resulting from the union of \( \text{obj}_1 \) and \( \text{obj}_2 \), can be defined as

\[
\text{obj}_{\text{new}} = \text{obj}_1 - \text{obj}_2
\]

The surface of the result is

\[
Surf(\text{obj}_1) - \text{obj}_2
\]

which can be expressed in terms of

\[
Surf(\text{obj}_1 \cup \text{obj}_2)
\]

This surface has, in general, the shape of points which satisfy \( f_1(x) \).

Based on transversality, discontinuity, and some conditions. Roughly, all contours except those lying in the boundary is a part boundary statement can be made about differential geometry, but the details are beyond the scope of this discussion. One further step is needed to convert surfaces such as the cosine patch of surface having a cosine smoothness slightly (e.g., by smoothing slightly (e.g., by

Clear that the concave disco
which can be expressed in terms of inside-outside functions as
\[
\text{Surf}(\text{obj}_1 \cup \text{obj}_2) = \{x \in \mathbb{R}^3 | f_1(x) = 0 \& f_2(x) > 0 \}
\]

\[
\text{OR}
\]

\[
f_1(x) > 0 \text{ OR } f_2(x) = 0
\]

This new composite surface has, in general, a contour of concave discontinuity, consisting of points which satisfy \( f_1(x) = f_2(x) = 0 \).

The new object formed by interpenetrating \( \text{obj}_1 \) with \( \text{obj}_2 \) and then removing \( \text{obj}_2 \) can be defined as the closed-set subtraction of \( \text{obj}_2 \) from \( \text{obj}_1 \).

\[
\text{obj}_{\text{new}} = \text{obj}_1 - \text{obj}_2 = \{x \in \mathbb{R}^3 | x \in \text{obj}_1 \& x \notin \text{obj}_2 \}
\]

\[
= \{x \in \mathbb{R}^3 | f_1(x) \leq 0 \& f_2(x) > 0 \}.
\]

The surface of the resulting object is then (Barr, 1983)

\[
\text{Surf}(\text{obj}_1 - \text{obj}_2) = \{x \in \text{Surf}(\text{obj}_1) \& x \notin \text{obj}_2 \}
\]

\[
\text{OR}
\]

\[
x \in \text{obj}_1 \& x \in \text{Surf}(\text{obj}_2)
\]

which can be expressed in terms of inside-outside functions as

\[
\text{Surf}(\text{obj}_1 - \text{obj}_2) = \{x \in \mathbb{R}^3 | f_1(x) = 0 \& f_2(x) > 0 \}
\]

\[
\text{OR}
\]

\[
f_1(x) < 0 \& f_2(x) = 0
\].

This surface has, in general, a contour of convex discontinuity, consisting of points which satisfy \( f_1(x) = f_2(x) = 0 \).

Based on transversality, then, we can take some contours of concave discontinuity and some contours of convex discontinuity to be part boundaries. Roughly, all contours of concave discontinuity are part boundaries except those lying in the bottom of a depression. And a contour of convex discontinuity is a part boundary only if it surrounds a depression. (This rough statement can be made precise and algorithmic using the language of differential geometry, but this is beyond the scope of this paper.)

One further step is needed to begin to define part boundaries on smooth surfaces such as the cosine surface of Figure 1. Consider what happens to a patch of surface having a concave discontinuity running through it if the patch is smoothed slightly (e.g., by draping a cotton sheet over it). Intuitively it is clear that the concave discontinuity will become a locus of very high curva-
ture, in fact a locus of points which are negative extrema of surface curvature in a suitable sense (see Figure 5 and the partitioning rule stated below). To give this intuition a rigorous proof, however, requires dealing with some technicalities of a differential geometric nature, and so a complete and careful analysis must be made. In §3 of this chapter we develop the necessary analytic framework and prove a fundamental theorem of the behavior of curvature as smooth surfaces "approach" a transversal intersection (Theorem 6). The theorem says that near this intersection curve there are points of arbitrarily large negative curvature on these smooth surfaces. To complete a rigorous justification of the intuition it must be shown that these points form contours—as in the rule stated below—on members of the family of smooth surfaces, and that these contours approach the intersection curve. This will be done in a subsequent paper.

Negative Minima Partitioning Rule: Divide a surface into parts whose boundaries are contours consisting of points which are negative minima of principal curvature along a line of curvature.

The boundaries defined by this negative minima rule are derived in §4 for several classes of surfaces. The resulting boundaries are sketched in several figures in that section so that one can better appreciate the effect of the rule. To avoid confusion, it should be noted that although the minima rule employs lines of curvature in its definition of part boundary, the part boundaries themselves are not, in general, lines of curvature.

The negative minima rule provides boundaries for all parts except depressions. Depressions are delimited by positive maxima of the principal curvatures along their associated lines of curvature, which is the rule one obtains by smoothing convex discontinuities. A further set of rules, which will not be described here, determine when the negative minima rule is to be used and

Figure 5. Smoothing a transversal intersection leading to negative minima of curvature, which in the figure occur along the curve in the shaded region

when, instead, the positive negative minima of the principal curvatures correspond, roughly, to the boundaries of a given part since they correspond to points of positive maxima of the principal curvatures.

This paper focuses entirely of the principal curvatures.

3. DERIVATION OF PARTITIONING RULES

In this section we will prove theorems which give large cases of surfaces which possess a "smoothing" of a given part boundary converging to it in the $C^0$ sense.

We begin with an example. Suppose we introduce in a concrete setting the proof, we consider in detail the case in which the set of points on the graph of $f(x, y) = xy = 0$, the so-called level set is precisely the set on which either $x = 0$ or $y = 0$.

Representing this transversal, we consider the convenient representation $g(x, y) = xy - \epsilon$, where $\epsilon$ is a small positive number. These functions $g(x, y)$ approach the case of transversal intersection as $\epsilon$ approaches zero. The family of level sets, with parameter $\epsilon$, connect as smooth parabolas.

The curvature, $k(x, \epsilon)$, on the curve to be:

$$k(x, \epsilon)$$
e extrema of surface curvature (partitioning rule stated below). To ensure dealing with some technique to a complete and careful development of the necessary analysis of the behavior of curvatures at an intersection (Theorem 6). We observe that points are points of arbitrariness. To complete a definition that these points form the family of smooth intersection curve. This will be a matter of a surface into parts whose which are negative minima of extrema rule are derived in §4 for minima and are sketched in several classical theorems the effect of the rule. Although the minima rule employs curvature, the part boundaries themselves for all parts except depressions of the principal curvatures which is the rule one obtains a set of rules, which will not be minima rule is to be used and when, instead, the positive maxima rule is to be used. Parts delimited by negative minima of the principal curvatures are called "positive parts" since they correspond roughly, to various kinds of bumps on an object. Parts delimited by positive maxima of the principal curvatures are called "negative parts" since they correspond to depressions in an object.

This paper focuses entirely on positive parts delimited by negative minima of the principal curvatures.

3. DERIVATION OF PARTITIONING RULES

In this section we will prove that smoothing a transversal intersection of surfaces leads to arbitrarily large negative curvature. It turns out that this is the case regardless of how the smoothing is accomplished in $C^2$, i.e., for our purposes a "smoothing" of a given surface will be a sequence of smooth surfaces converging to it in the $C^2$ sense; for precise definitions see below.

We begin with an example intended to make plausible the claim and to introduce in a concrete setting several concepts used in the proof. Following the proof, we consider in detail two special cases of smoothing which it subsumes: (1) smoothing with a Gaussian and (2) smoothing by spline approximation.

3.1 Smoothing transversal intersections: An example

One particularly simple example of a transversal intersection is that formed by the two lines $y = 0$ and $x = 0$, i.e., by the $x$ and $y$ axes, as shown in Figure 6a. Consider the sets of points in the plane which satisfy the equation $f(x, y) = xy = 0$, the so-called "zero level set" of the function $f(x, y)$. This level set is precisely the desired two lines, because it is the set of points on which either $x = 0$ or $y = 0$.

Representing this transversal intersection by means of a level set leads to a convenient representation for smoothing. Consider the set of functions $g(x, y) = xy - \varepsilon$, where $\varepsilon \geq 0$. As $\varepsilon$ approaches zero, the zero level sets of these functions $g(x, y)$ approach the zero level set of $f(x, y)$, i.e. they approach the case of transversal intersection, as shown in Figure 6b. In effect, $\varepsilon$ serves as a smoothing parameter, with larger values of $\varepsilon$ indicating a greater degree of smoothing. The parameter $\varepsilon$ can also be thought of as an index into the family of level sets, with each value of $\varepsilon$ uniquely associated with one level set.

The curvature, $k(x, \varepsilon)$, on these level sets can be found by standard formulae to be:

$$k(x, \varepsilon) = \frac{-2\varepsilon x^{-3}}{(\sqrt{1 + x^{-4}})^3}$$
For a particular choice of \( \varepsilon \), i.e., for any particular member of the family of level sets, the curvature will have its greatest absolute value (and negative sign) at the point where the level set intersects the line \( y = x \). This can be seen by noting the symmetry of the level sets about the line \( y = x \) in Figure 6b. Now along this line we have that \( x = y = \varepsilon x \), so that \( x = \sqrt{\varepsilon} \). Substituting this relation into the equation for curvature, and simplifying, we find that the negative minimum of curvature for the level set \( \varepsilon \) is

\[
\kappa_{\min}(\varepsilon) = 1/2\varepsilon.
\]

Now as \( \varepsilon \to 0 \), i.e., as the level sets approach the singular level set, the minimum of curvature goes to \(-\infty\). Thus we see intuitively that smoothing the transversal intersection by means of this family of level sets replaces the singular point with negative minima of curvature, as illustrated by Figure 6b.

### 3.2 Preliminaries on the curvature of level sets

The previous section demonstrated, for a simple example, that smoothing a transversal intersection leads to negative minima of curvature. In this section we begin the proof that this result holds for all transversal intersections and all smoothings.

We start by considering surface curvature. Curvature is a priori a property of a surface (or manifold) at a point. However, in most applications the surface in question is naturally defined as the level set of a function. Thus if \( f \) is a function on a domain \( D \), through \( P \), i.e. the set

\[
M(f, P)
\]

We note that there are through \( P \). For example \( f_1 = hf_2 \), where \( h \) is a nity, our point of view is defined by particular functions, practice, and also leads to which is our ultimate interest.

The most general class of admissible notions of curvature are partial derivatives through which denotes the set of functions to be the set of such functions together with their derivations, and extensions to the boundary, where we can define the metric

\[
\| f_1 - f_2 \|
\]

where \( \partial \) ranges over all par nate systems of order \( 0 \) through \( P \).

We now consider a dom: the level set \( M(f, P) \). If \( \nabla \) smooth surface through \( P \): \((x, y, z) \) so that \( \nabla f(P) \) po the origin, as shown in fig 6C function \( g(x, y) \) on a near \( P \), \( M(f, P) \) is the \( P = (0,0) \),

resembles the so called “g and its eigenvalues are by denoted by \( k_1(f, P), k_2(f, P) \). we may assume that \( g_{xy}(P) = 0 \).

\[
\begin{pmatrix}
g_{xx}(P) & g_{xy}(P) \\
0 & g_{yy}(P)
\end{pmatrix}
\]
a function on a domain \( D \), and if \( P \in D \), we can look at the level set of \( f \) through \( P \), i.e., the set

\[
M(f, P) = \{ Q \in D \mid f(Q) = f(P) \}.
\]

We note that there are many functions which have the same level set through \( P \). For example the sets \( f_1 = 0 \) and \( f_2 = 0 \) are the same if \( f_1 = h f_2 \), where \( h \) is a nowhere vanishing function. In spite of this ambiguity, our point of view here will be that of surfaces in \( \mathbb{R}^3 \) as level sets defined by particular functions. This reflects the situations which arise in practice, and also leads most naturally to the study of variation of level sets, which is our ultimate interest in this section.

The most general class of functions on whose level sets there is a reasonable notion of curvature are the "\( C^2 \) functions", i.e., functions with continuous partial derivatives through the second order. If \( D \) is a domain in \( \mathbb{R}^n \), \( C^2(D) \) denotes the set of functions on \( D \). If \( D \) is compact then \( C^2(D) \) can be defined to be the set of such functions which are \( C^2 \) on the interior of \( D \), and which, together with their derivatives through the second order, have continuous extensions to the boundary of \( D \). With this definition \( C^2(D) \) is a metric space, where we can define the metric, \( \| \cdot \|_{C^1} \), as follows: Let \( f_1, f_2 \in C^2(D) \). Then

\[
\| f_1 - f_2 \|_{C^1} = \sup_{P \in D} \{ |\partial f_1(P) - \partial f_2(P)| \},
\]

where \( \partial \) ranges over all partial derivatives (with respect to some fixed coordinate system) of order \( 0 \) through \( 2 \).

We now consider a domain \( D \subset \mathbb{R}^3 \), and let \( f \in C^2(D) \). For \( P \in D \), we have the level set \( M(f, P) \). If \( \nabla f(P) \) (the gradient of \( f \) at \( P \)) \( \neq 0 \), \( M(f, P) \) is a smooth surface through \( P \); we can choose an orthogonal coordinate system \( (x, y, z) \) so that \( \nabla f(P) \) points in the direction of the positive \( z \)-axis, and \( P \) is the origin, as shown in figure 7. By the implicit function theorem there is a \( C^2 \) function \( g(x, y) \) on a neighborhood of the origin in the \( x, y \)-plane so that near \( P \), \( M(f, P) \) is the graph \( z = g(x, y) \). The Hessian matrix of \( g \) at \( P = (0,0) \),

\[
\begin{pmatrix}
ge_{xx}(P) & g_{xy}(P) \\
ge_{yx}(P) & g_{yy}(P)
\end{pmatrix},
\]

represents the so-called "second fundamental form" of the surface \( M_f \) at \( P \), and its eigenvalues are by definition the principal curvatures of \( M(f, P) \) at \( P \), denoted by \( \kappa_1(f, P), \kappa_2(f, P) \). Thus, after a suitable rotation of the \( xy \)-plane, we may assume that \( g_{xy}(P) = 0 \), so that the Hessian of \( g \) at \( P \) is now

\[
\begin{pmatrix}
ge_{xx}(P) & 0 \\
0 & g_{yy}(P)
\end{pmatrix} = \begin{pmatrix}
\kappa_1(f, P) & 0 \\
0 & \kappa_2(f, P)
\end{pmatrix}.
\]
Figure 7. A level set $M$, with an orthogonal coordinate system centered at some point $P$ of $M$.

Now, from the relation $f(x, y, g(x, y)) = \text{constant}$, we deduce

$$f_x(x, y, g) + f_z(x, y, g)g_x = 0, \quad f_y(x, y, g) + f_z(x, y, g)g_y = 0. \quad (3.1)$$

Using the fact that $\nabla f(P) = (0, 0, |\nabla f(P)|)$, i.e., that $f_x(P) = f_y(P) = 0$ and $f_z(P) \neq 0$, the equations (3.1) imply that $g_x(0, 0) = g_y(0, 0) = 0$. This, together with an additional differentiation of the equations (3.1) with respect to $x$ and $y$ yields:

$$f_{xx}(P) + f_z(P)g_{xx}(0, 0) = 0, \quad f_{yy}(P) + f_z(P)g_{yy}(0, 0) = 0,$$

that is,

$$\kappa_1(f, P) = \frac{f_{xx}(P)}{\nabla f(P)}, \quad \kappa_2(f, P) = \frac{f_{yy}(P)}{\nabla f(P)},$$

and finally:

$$\kappa_1(f, P) = \frac{f_{xx}(P)}{\sqrt{f}(P)}, \quad \kappa_2(f, P) = \frac{f_{yy}(P)}{\sqrt{f}(P)}. \quad (3.2)$$

This expression depends on the particular $x$-$y$-$z$ coordinate system which is associated as above to $f$ and $P$. We want to express this same relation in a form which is coordinally surfaces of $f$ with the Hessians of $f$ on $R^3$, which can be done in the system, say $(u^1, u^2, u^3)$ with coordinates $f_{u^i u^j}(P)$. These elements, is independent of $\text{tr}H(f, P)$.

Since in terms of $\text{tr}H(f, P) = f_{xx}(P) + f_{yy}(P) + f_{zz}(P)$,

$$\kappa_1(f, P) + \kappa_2(f, P)$$

Here $z$ itself has an important role as a normal vector $\nabla f(p)/|\nabla f(p)|$.

Thus we may rewrite (3)

**Proposition 1:** If $f$ forms $\omega$ a matrix $A$ and $N(f, P)$, we may write

$$f_{zz}(P)$$

(This means that in the

$$\text{tr}H(f, P) = f_{xx}(P) + f_{yy}(P) + f_{zz}(P)$$

where $\kappa_i(f, P)$ are the $H(f, P)$ is the Hessian $M(f, P)$ at $P$, i.e. $N(f, P)$.

**Remark:** The mean curvature" of the "mean curvature" of the $\omega$ field.

### 3.3 Level sets with $\text{tr}H(f, P)$

As above, let $D$ be a $\text{tr}H(f, P)$ be the level sets for $f$, $B_1$ and $B_2$, which are shown in Figure 8.

The two smooth surface loci of $M$. The traces at each point $P_0$ of $S$, etc., $R^3$, i.e. the tangent plane intersect in the tangent
form which is coordinate free, so that we can compare the curvatures of the level surfaces of $f$ with those of “nearby” functions. For this purpose we consider the Hessian of $f$ itself at $P$, denoted $H(f, P)$. This is a quadratic form on $\mathbb{R}^3$, which can be defined intrinsically. In any given orthogonal coordinate system, say $(u^1, u^2, u^3)$, it is represented by the matrix of second partial derivatives $f_{u^i u^k}(P)$. The trace of this matrix, i.e. the sum of its diagonal elements, is independent of the particular coordinate system; we will denote it by $\text{tr}H(f, P)$.

Since in terms of the $x$-$y$-$z$ system discussed above we have $\text{tr}H(f, P) = f_{xx}(P) + f_{yy}(P) + f_{zz}(P)$, the equations (3.2) imply:

$$\kappa_1(f, P) + \kappa_2(f, P) = \frac{\text{tr}H(f, P) - f_{zz}(P)}{|\nabla f(P)|}.$$

Here $z$ itself has an intrinsic meaning as an axis in the direction of the unit normal vector $\nabla f(P)/|\nabla f(P)|$ to $M(f, P)$ at $P$. If we denote this unit vector $N(f, P)$, we may write

$$f_{zz}(P) = N(f, P)^tH(f, P)N(f, P).$$

This means that in the given coordinate system, if the Hessian is represented by a matrix $A$ and $N(f, P)$ by a column vector $B$, then $f_{zz}(P) = B^tA^tB$.

Thus we may rewrite (3.3), and we get:

**Proposition 1:** If $f$ is $C^2$ around $P \in \mathbb{R}^3$, and $\nabla f(P) \neq 0$, then

$$\kappa_1(f, P) + \kappa_2(f, P) = \frac{\text{tr}H(f, P) - N(f, P)^tH(f, P)N(f, P)}{|\nabla f(P)|},$$

where $\kappa_1(f, P)$ are the principal curvatures of the level set $M(f, P)$ at $P$, $H(f, P)$ is the Hessian form of $f$ at $P$, and $N(f, P)$ is the unit normal to $M(f, P)$ at $P$, i.e. $N(f, P) = \nabla f(P)/|\nabla f(P)|$.

**Remark:** The quantity $\kappa_1(f, P) + \kappa_2(f, P)$ is sometimes called the “mean curvature” of the surface $M(f, P)$ at $P$; we will denote it $\mu(f, P)$.

### 3.3 Level sets with transversal intersections

As above, let $D$ be a domain in $\mathbb{R}^3$, and suppose that $\phi \in C^2(D)$ has the following property: the level set $M: \phi = 0$ consists locally of two smooth surfaces, $B_1$ and $B_2$, which intersect transversally along a smooth curve $S$, as shown in Figure 8.

The two smooth surfaces are called the “branches” of $M$; $S$ is the “singular locus” of $M$. The transversality of the intersection of the branches means that at each point $P_0$ of $S$, the union of the tangent spaces to the branches generates $R^3$, i.e. the tangent planes are not parallel, and the two tangent spaces intersect in the tangent space to $S$.
Figure 8. A level set consisting of two smooth branches intersecting transversally

\[ T_{P_0}(B_1) \cap T_{P_0}(B_2) = T_{P_0}(S) \]

\[ T_{P_0}(B_1) + T_{P_0}(B_2) = \mathbb{R}^2. \]

Given such a \( \phi \), and \( P_0 \in S \), we will choose an orthogonal coordinate system \((u, v, w)\) on \( \mathbb{R}^3 \) that \( P_0 = (0, 0, 0) \), the \( u \)-axis is tangent to \( S \), and the \( v \) and \( w \) axes are chosen as follows: Let \( q_1 \) be a unit vector along the \( u \)-axis, i.e. \( q_1 \in T_{P_0}(S) \). For \( i = 1, 2 \) let \( r_i \in T_{P_0}(B_i) \) with \( r_i \perp q_1 \) and \( |r_i| = 1 \). Let \( q_3 \) be \((r_1 + r_2)/|r_1 + r_2|\), i.e. \( q_3 \) is a unit “bisector” of \( r_1 \) and \( r_2 \). Let \( q_2 \) be \( q_3 \times q_1 \). Finally choose the coordinates \( v \) and \( w \) so that the positive \( v \) and \( w \) axes are in the \( q_2 \) and \( q_3 \) directions respectively. The picture is shown in Figure 9.

Figure 9. A canonical coordinate system associated with the level surface \( M; \psi = 0 \).

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Remark: Note that the characteristic directions \( q_1, q_2, q_3 \) are orthogonal to the tangent space of \( S \) at \( P_0 \), and that they form a basis for \( \mathbb{R}^3 \).

If we intersect this figure \( \langle u = 0 \rangle \) with \( B_i \), any \( i \), we get a system of characteristic directions. Thus at \( P_0 \), we have, near \( P_0 = (0, 0, 0) \),

\[ b_i(u, v, w) = \frac{\partial b_i}{\partial u}(P_0) + \beta_i \frac{\partial b_i}{\partial u}(P_0) + \cdots \]

where \( \beta_i \) is a \( C^2 \) function whose order \( \beta_i \) is a polynomial in \( u \) and \( v \), and that of its first \( \beta_i \) is the common tangent space at \( P_0 \) is the null space of space of
If we intersect this figure with the $v$-$w$ plane, we get Figure 10.

**Remark:** Note that the choices of orientation of $r_1$ and $r_2$ give rise to four possible values of $q_3$, any two of which are either orthogonal or point in opposite directions. Thus alternate choices may reverse the roles of $q_2$ and $q_3$, or reverse their orientations. In the analysis of a shape from the point of view of a given observer, however, there will generally be a "natural" choice of $q_3$ which is "visible" to the observer.

Let $b_i$ be any $C^2$ function whose 0 level set is $B_i$, i.e. $B_i$: $b_i = 0$. In the $u$, $v$, $w$ system $b_i(P_0) = b_i(0, 0, 0) = 0$. Therefore, by Taylor's theorem, we have, near $P_0(0, 0, 0)$,

$$b_i(u, v, w) = \alpha_i u + \beta_i v + \gamma_i w + \epsilon_i(u, v, w),$$

where $\alpha_i = \partial b_i / \partial u(P_0)$, $\beta_i = \partial b_i / \partial v(P_0)$, $\gamma_i = \partial b_i / \partial w(P_0)$. $\epsilon_i(u, v, w)$ is a $C^2$ function whose order of vanishing at $P_0$ is greater than 1, i.e. whose values, and that of its first partial derivatives, is 0 at $P_0$. Now the intersection locus $S$ is the common set of zeroes $b_1 = b_2 = 0$; the tangent space to $S$ at $P_0$ is the null space of the Jacobian matrix of $b_1$, $b_2$ at $P_0$, i.e. the null space of

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_1 \end{pmatrix}$$

Figure 10. The intersection of Figure 8 with the $v$, $w$-plane
in the \(u, v, w\) system. However this system was chosen so that the tangent space to \(S\) is the \(u\)-axis, from which it follows that \(\alpha_1 = \alpha_2 = 0\). Moreover, the unit tangent vectors to \(B_1 \cap \{u = 0\}\), i.e. \(r_1, r_2\) in Figure 10 above, are reflections of each other about the \(w\)-axis, and it follows from this that \((\beta_2, \gamma_2) = k(-\beta_1, \gamma_1)\), for some constant \(k\). Thus we may assume

\[
\begin{align*}
b_1 &= \beta v + \gamma w + \epsilon_1(u, v, w), \\
b_2 &= -\beta v + \gamma w + \epsilon_2(u, v, w),
\end{align*}
\]

for suitable numbers \(\beta, \gamma\).

But then, near \(P_0, B_1 \cup B_2\) is the level set \(b = 0\), where

\[
b = -\beta^2 v + \gamma^2 w + \epsilon(u, v, w),
\]

(and now \(\epsilon\) is a function which vanishes at \(P_0\) to order greater than 2).

Now since \(\phi\) and \(b\) have the same zero locus, there exists a non-vanishing function \(h\) so that \(\phi = hb\). At \(P_0 = (0, 0, 0)\), \(h(u, v, w) = k + \) higher order terms in \((u, v, w)\) where \(k \in \mathbb{R}, k \neq 0\). Thus, near \(P_0\),

\[
\phi(u, v, w) = -k\beta^2 v + k\gamma^2 w + E(u, v, w),
\]

where \(E(u, v, w)\) is \(C^2\), and vanishes at \(P_0\) to order greater than 2. Letting \(l = k\beta^2, m = k\gamma^2\), we can summarize as follows:

**Proposition 2:** Let \(\phi\) be a \(C^2\) function on a domain \(D\) in \(\mathbb{R}^3\), and suppose \(M: \phi = 0\) is the union of two smooth surfaces (the "branches") which intersect in a smooth curve \(S\). Then there exists a right-handed orthogonal coordinate system \(u, v, w\) in which \(P_0\) is the origin, and near \(P_0\),

\[
\phi(u, v, w) = -lv^2 + mw^2 + E(u, v, w),
\]

where:

(a) \(l, m\) are distinct positive numbers such that the tangent planes to the branches are \(\sqrt{l}v + \sqrt{m}w = 0\) and \(-\sqrt{l}v + \sqrt{m}w = 0\) respectively,

(b) \(E(u, v, w)\) is a \(C^2\) function which vanishes at \(P_0\) to order greater than 2, i.e. its partial derivatives through the second order are zero at \(P_0\),

(c) the \(u, v, w\) coordinate system is uniquely determined up to a rotation through a multiple of \(\pi/2\) in the \(v\) \(w\) plane (corresponding to the four possible choices of \(q_3\) as in the remark above), and reversal of direction of the \(w\)-axis.

We will refer to this expression of \(\phi\) in the \(u, v, w\) as the canonical presentation of \(\phi\) at \(P_0\); the \(u, v, w\) system itself is called the canonical coordinate system.

We now want to study the nearby level surfaces \(\phi = t\), and especially certain features of their behavior at \(t \to 0\). At any point \(P\), we can consider the level set \(M(\phi, P)\) of \(\phi\) through \(P\) and as \(P \to P_0\), \(M(\phi, P) \to M(\phi, P_0) =

**M.** Let us restrict our attention to smooth; by Sard's theorem (position 2), it is seen that if \(\gamma\) is smooth, there is a \(w\) where \(N(\phi, P) = \nabla \phi(P)/|\nabla \phi|\) ever, a natural question (and \(\lim(\phi, P)\) exists).

**Proposition 3:** Let \(\gamma\) be a tangent vector \((a, b, c)\) at \(\phi\). Then

\[
\lim_{P \to P_0} N(\phi, P)
\]

Thus, the limit depends only on \(P_0\), provided that this direction is.

**Proof:** Assume we have a system \(u, v, w\). Let \(\gamma\) with unit tangent vector \(\gamma(s) = (as + A(s), bs + \) vanish at \(s = 0\) to order at \(Now \nabla \phi(u, v, w) = (E_v, E_w, E_w / E_v)\),

\[
\nabla \phi(\gamma(s)) = \left[E_v(\gamma(s)), -2lb s\right]
\]

\[
= s \left(0, -2lb, 2mc\right) + \left[E_v(\gamma(s))\right]
\]

2mC(s) + E_w(\gamma(s))/s\) is \(v\) and \(C(s)\) are actually divisible \(s\), so it is clear that these \(v\) partials of \(E\) at \((0, 0, 0)\) \(w\) and \(v\) position 2 above), the partials we may use.

**Lemma:** Let \(G\) be a \(C^2\) \(G_v\) all vanish at \((0, 0, 0)\) curve \(\gamma\) with \(\gamma(0) = 0\).

**Proof of Lemma:**

\[
\lim_{s \to 0} \frac{G(\gamma(s))}{s} = \lim_{s \to 0} \frac{G(\gamma(s))}{s}
\]
was chosen so that the tangent that $\alpha_1 = \alpha_2 = 0$. Moreover, $r_1, r_2$ in Figure 10 above, are and it follows from this that Thus we may assume

$$u, v, w,$$

$$v = 0,$$

where

$$u, v, w,$$

0 order greater than 2).

as, there exists a non-vanishing 0), $b(u, v, w) = k +$ higher thus, near $P_0$,

$$+ E(u, v, w),$$

3 order greater than 2. Letting its domain $D$ in $\mathbb{R}^3$, and suppose $s$ (the "branches") which intertwined-handed orthogonal coordinate near $P_0$

$$+ E(u, v, w),$$

such that the tangent planes to $Tv + \sqrt{m}w = 0$ respectively, at $P_0$ to order greater than 2, order are zero at $P_0$, and (c) the need up to a rotation through a $\pi$ to the four possible choices of section of the $w$-axis.

$v, w$ as the canonical present called the canonical coordinate

asses $\phi = t$, and especially certain point $P$, we can consider the $P_0, M(\phi, P) \rightarrow M(\phi, P_0) = M$. Let us restrict our attention to those $P$ near $S$ for which $M(\phi, P)$ is smooth; by Sard’s theorem (or using the canonical representation of $\phi$ of Proposition 2), it is seen that there is a neighborhood of $S$ on which this is true. Thus, for $P \in S$, there is a well defined unit normal $N(\phi, P)$ to $M(\phi, P)$ at $P$; here $N(\phi, P) = \nabla \phi(P)/|\nabla \phi(P)|$. Now as $P \rightarrow P_0$, $|\nabla \phi(P)| \rightarrow 0$. However, a natural question (and one which is important for the sequel) is whether

$$\lim_{P \rightarrow P_0} N(\phi, P)$$

exists.

**Proposition 3:** Let $\gamma$ be a differentiable curve through $P_0$, with unit tangent vector $(a, b, c)$ at $P_0$ (in the canonical $u, v, w$ coordinate system for $\phi$). Then

$$\lim_{P \rightarrow P_0} N(\phi, P) = \frac{(0, -lb, mc)}{(0, -lb, mc)}, \quad P \in \gamma.$$ 

Thus, the limit depends only on the tangent direction of $\gamma$ at $P_0$, and it exists provided that this direction is not that of the $u$-axis, i.e., $\gamma$ is not tangent to $S$.

**Proof:** Assume we have a canonical presentation of $\phi$ at $P_0$ with coordinate system $u, v, w$. Let $\gamma$ be a differentiable curve through $P_0 = (0, 0, 0)$, with unit tangent vector $(a, b, c)$ at $P_0$. Thus if $s$ is arclength, then

$$\gamma(s) = (as + A(s), bs + B(s), cs + C(s)),$$

where $A(s), B(s)$, and $C(s)$ vanish at $s = 0$ to order at least 2 (i.e., $A(s), B(s), C(s)$ are divisible by $s^2$).

Now $\nabla \phi(u, v, w) = (E_u, -2lv + Ev, 2mv + Ew)$, so that

$$\nabla \phi(\gamma(s)) = [E_u(\gamma(s)), -2bs - 2lb(s) + E_u(\gamma(s)), 2mcs + 2mc(s) + E_u(\gamma(s))]$$

$$= s \left[ (0, -2lb, 2mc) + \left( \frac{E_u(\gamma(s))}{s}, \frac{-2lb(s) + E_u(\gamma(s))}{s}, \frac{-2mc(s) + E_u(\gamma(s))}{s} \right) \right]$$

Let us denote by $F(s)$ the vector $(E_u(\gamma(s))/s, -2lb(s) + E_u(\gamma(s))/s, 2mcs + 2mc(s) + E_u(\gamma(s))/s)$. We claim that $F(s)$ vanishes as $s \rightarrow 0$. Since $B(s)$ and $C(s)$ are actually divisible by $s^2$, $B(s)/s$ and $C(s)/s$ are still divisible by $s$, so it is clear that these vanish. For $E_u/s$, $E_v/s$, $E_w/s$, we note that since the partials of $E$ at $(0, 0, 0)$ vanish through the second order (by part (b) of Proposition 2 above), the partials of $E_u$, $E_v$, $E_w$ vanish through the first order, so we may use

**Lemma:** Let $G$ be a $C^2$ function around $(0, 0, 0)$ and suppose $G, G_u, G_v, G_w$ all vanish at $(0, 0, 0)$. Then $\lim_{s \rightarrow 0} G(\gamma(s))/s = 0$ for any differentiable curve $\gamma$ with $\gamma(0) = 0$.

**Proof of Lemma:**

$$\lim_{s \rightarrow 0} \frac{G(\gamma(s))}{s} = \lim_{s \rightarrow 0} \frac{G(\gamma(s)) - G(\gamma(0))}{s} = \frac{d}{ds} G(\gamma(s)) |_{s=0}.$$
But this derivative may also be computed in the form
\[
\frac{d}{ds} G(\gamma(s)) = G_s(\gamma(s)), \quad G_v(\gamma(s)), \quad G_w(\gamma(s)) \gamma'(0),
\]
and this is zero by hypothesis.

Returning to the proof of Proposition 3, we have
\[
\nabla \phi(\gamma(s)) = s[(0, 2lb, 2mc) + F(s)],
\]
where \(\lim_{s \to 0} F(s) = (0, 0, 0)\). Hence
\[
\| \nabla \phi(\gamma(s)) \| = |s| \left(\| (0, -2lb, 2mc) \| + h(s) \right),
\]
where \(h(s) \to 0\) as \(s \to 0\), as is easily seen. Therefore
\[
\frac{\nabla \phi(\gamma(s))}{\| \nabla \phi(\gamma(s)) \|} = \pm \frac{(0, -2lb, 2mc) + F(s)}{\| (0, -2lb, 2mc) \| + h(s)},
\]
so that
\[
\lim_{s \to 0} \frac{\nabla \phi(\gamma(s))}{\| \nabla \phi(\gamma(s)) \|} = \pm \frac{(0, -2lb, 2mc)}{\| (0, -2lb, 2mc) \|}, \quad \text{i.e.}
\]
\[
\lim_{s \to 0} N(\phi, \gamma(s)) = \pm \frac{(0, -lb, mc)}{\| (0, -lb, mc) \|}.
\]
Note that the sign on the right will be + when \(s'/|s'| = 1\), i.e. when \(P \to P_0\) from the positive direction with respect to the parametrization of \(\gamma\). To obtain the result for approach to \(P_0\) along \(\gamma\) from the other direction, simply reverse the orientation of the arc-length parametrization. This completes the proof of proposition 3.

To introduce the basic idea of our main theorem which will be treated in detail in the next section, we now state a prototype of this result, which describes the behavior of the curvature of the level surfaces of \(\phi\) itself at points \(P\) approaching \(P_0\).

**Proposition 4:** With notation and hypotheses as above, let \(\gamma\) be a differentiable curve whose unit tangent vector, say \((a, b, c)\) in the \(u, v, w\) system, is not contained in the tangent planes to the branches \(B_i\) of \(\phi = 0\) at \(P_0\), and is in the same sector formed by these planes as the positive \(w\)-axis. For any \(P \in S\) let \(\mu(\phi, P)\) denote the mean curvature of the level surface \(M(\phi, P)\) of \(\phi\) through \(P\).\(^1\) (Thus \(\mu(\phi, P) = \kappa_1(\phi, P) + \kappa_2(\phi, P)\).) Then
\[
\lim_{P \to P_0} \mu(\phi, P) = -\infty.
\]

\(^1\) It is understood that we are restricting ourselves to a neighborhood of \(P_0\) in which the only non-smooth level surface of \(\phi\) is \(\phi = 0\), i.e. it is only on \(S\) that \(\nabla \phi\) vanishes.

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**Proof:** By Proposition 1,
\[
\mu(\phi, P) = \frac{\text{tr} H(\phi, P)}{\| \nabla \phi(\gamma(s)) \|},
\]
This may be computed in an system for convenience. The tion of \(P\), and
\[
H(\phi, P),
\]
Hence,
\[
\lim_{P \to P_0} \text{tr} H(\phi, P) = -\frac{1}{P - P_0},
\]
is the unit vector \((0, -lb, m)
\[
\lim_{P \to P_0} N(\phi, P) H(\phi, P) N(\phi, P),
\]
\[
= (Pb^2 + \frac{1}{P - P_0})
\]
Therefore, \((*)\) the limit of th is
\[
-1 + \frac{1}{P - P_0}
\]
which simplifies to \((-mc^2\).
the form

\[(s), \mathcal{G}_w(\gamma(s))\gamma'(0),\]

we have

\[b, (2mc) \| \mathbf{h}(s)\|,\]

wherefore

\[b, (2mc) + \mathbf{h}(s),\]

therefore

\[b, (2mc) + \mathbf{h}(s),\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & -l & 0 \\
0 & 0 & m
\end{pmatrix}
\]

Hence, \(\lim_{P \to P_0} N(\phi, P) = \mathbf{h}(s)\). By Proposition 3

\[
\lim_{P \to P_0} N(\phi, P) = \mathbf{h}(s)
\]

is the unit vector \((0, -lb, mc) \sqrt{l^2b^2 + m^2c^2}\). By continuity, we obtain

\[
\lim_{P \to P_0} N(\phi, P) = (l^2b^2 + m^2c^2)^{-1} \begin{pmatrix}
0 & 0 & 0 \\
0 & -l & 0 \\
0 & 0 & m
\end{pmatrix} \begin{pmatrix}
0 \\
lb \\
mc
\end{pmatrix} = \frac{-l^3b^2 + m^3c^2}{l^2b^2 + m^2c^2}.
\]

Therefore, \((\ast)\) the limit of the numerator in the expression for \(\mu(\phi, P)\) above is

\[
-l + m - \frac{-l^3b^2 + m^3c^2}{l^2b^2 + m^2c^2},
\]

which simplifies to \((-mc^2 + lb^2)d\) where \(d = (l^2b^2 + m^2c^2)^{-1} > 0\). Setting this equation to 0, and solving for \(b\) and \(c\), we get \(-mc^2 + lb^2 = 0\). The set of vectors \((a, b, c)\) for which this relation holds is the union of the planes \(-\sqrt{l}b + \sqrt{mc} = 0\) and \(\sqrt{l}b + \sqrt{mc} = 0\), i.e., the tangent planes to the branches of \(\phi = 0\) at \(P_0\) as shown in Figure 11. Similarly, the set of vectors \((a, b, c)\) which yield a negative numerator in the limit are those...
Cross-section through $P_0$ (in the $v$, $w$-plane) of the tangent planes to the branches of $M$ at $P_0$; the limit of the mean curvatures, $\mu(\psi, P)$, as $P \to P_0$ along $\gamma$ in the shaded sector as shown, is negative because of the orientation of our canonical coordinate system.

Figure 11.

$(a, b, c)$ for which $|c/b| > \sqrt{1/m}$, i.e. those which lie on the same sector (formed by the tangent planes to the branches) as the positive $w$-axis.

Thus let $\gamma$ be a curve satisfying the hypothesis. As $P \to P_0$ along $\gamma$ the limit of the numerator in the expression for $\mu(\phi, P)$ is a negative number. Since $\lim_{P \to P_0} \|\nabla \phi\| = 0$, i.e. since the denominator approaches 0 through positive values, we get the result.

We now want to study the question of uniformity of approach of $\mu(\phi, P)$ to $-\infty$ as $P \to P_0$, in suitably restricted regions:

Remark: Let $k > \sqrt{1/m}$, and consider the solid cone $C$ in $R^3$ with vertex $P_0$, directrix the positive $w$-axis, and slope $k$, as shown in Figure 12.

Figure 12. Restricting our attention will enable us to conclude that through $P$, goes to $-\infty$ as $P \to P_0$.

Note that, except at $P_0$, the $w = \pm \sqrt{1/m}$, i.e. except at the origin formed by these planes.

Consider the set $V$ of all uncurved with $\gamma(P_0) = v$, then the expression for $\mu(\phi, P)$ in $v$. Moreover, from the explicit proof of Proposition 4, it is clear that compact, there is a maximum $v$ must still be strictly negative f there must be a vector in $V$ branches of $\phi = 0$, and this co
Figure 12. Restricting our attention to points $P$ in cones of the type of $C$ in this figure will enable us to conclude that the mean curvature at $P$ of the level surface of $\psi$ through $P$, goes to $-\infty$ as $P \to P_0$.

Note that, except at $P_0$, the boundary of this cone does not meet the planes $w = \pm \sqrt{\sqrt{3}} v$, i.e. except at $P_0$ it is strictly contained in the upper sector formed by these planes.

Consider the set $V$ of all unit vectors lying in $C$. For each $v \in V$, if $\gamma$ is a curve with $\gamma(P_0) = v$, then the limit as $P \to P_0$ along $\gamma$ of the numerator of the expression for $\mu(\phi, P)$ in Proposition 4 is negative, and depends only on $v$. Moreover, from the explicit expression for this numerator derived in the proof of Proposition 4, it is clear that it depends continuously on $v$. Since $V$ is compact, there is a maximum value $\mu$ for this numerator attained on $V$, and it must still be strictly negative for otherwise (as in the proof of Proposition 4) there must be a vector in $V$ which lies in the tangent plane to one of the branches of $\phi = 0$, and this contradicts the construction of $V$. 

\begin{align*}
\text{tangent planes to} \\
\text{branches of } \phi
\end{align*}
Now for $P \in C$, consider the line joining $P_0$ with $P$, viewed as a curve $\gamma$ parametrized so that $\gamma(0) = P_0$ and $P \in \gamma$. Let $n(P)$ denote the value of the numerator of $\mu(\psi, P)$, and let $\bar{n}(P)$ denote its limit as $P \to P_0$ along this $\gamma$. Since $\bar{n}(P)$ depends only on $\gamma(P_0)$, i.e., in this case on the unit vector in the direction $\overrightarrow{P_0P}$, it is clear that $\bar{n}(P)$ is a continuous function of $P$ and that $\bar{n}(P) \leq \mu < 0$ for all $P \in C$. If $\overline{C}$ denotes the truncation of $C$ at some convenient value of $w$, then $\overline{C}$ is compact, and the function $\bar{n}(P)$ restricted to $\overline{C}$ is therefore uniformly continuous. Moreover, the function $n(P)$ itself is continuous in $P$, so $n$ is also uniformly continuous on $\overline{C}$.

Consider for the moment points $Q$ on the $w$-axis; for all these points $\bar{n}(Q)$ is the same (and in fact equals $-\bar{\lambda}$). There exists an $\varepsilon_1 > 0$ such that if $|Q - P_0| < \varepsilon_1$, then $|n(Q) - \bar{n}(Q)| < |\mu|/4$. By the uniform continuity of $n$, there exists $\varepsilon_2$ so that for all $P, P' \in \overline{C}$, $|P - P'| < \varepsilon_2 \Rightarrow |n(P) - n(P')| < |\mu|/4$. Let $\varepsilon_0$ be sufficiently small so that $\varepsilon_0 < \varepsilon_1$, and moreover that the cross-section of $\overline{C}$ slant height $\varepsilon_0$ has radius less than $\varepsilon_2$ (see Figure 13). Note that then if $P \in \overline{C}$ with $|P - P_0| < \varepsilon_0$, then if $Q$ denotes the point on the $w$-axis with the same $w$-coordinate as $P$, we have both $|P - Q| < \varepsilon_2$ and $|Q - P_0| < \varepsilon_1$.

It follows that

$$|n(P) - \bar{n}(Q)| \leq |n(P) - n(Q)| + |n(Q) - \bar{n}(Q)| \leq \frac{|\mu|}{4} + \frac{|\mu|}{4} = \frac{|\mu|}{2}. $$

Since $\bar{n}(Q) \leq \mu$, we find: There exists $\varepsilon_0$ so that $P \in \overline{C}$, $|P - P_0| < \varepsilon_0 \Rightarrow n(P) \leq \mu/2$ ( $\mu$ a fixed negative number).

Now since $\phi$ is $C^2$, $\|\phi(P)\| = 0$ uniformly on $\overline{C}$ as $P \to P_0$. It follows that: $\mu(\phi, P) = -\infty$ uniformly on $\overline{C}$ as $P \to P_0$. We may summarize this in the following result which may now be stated in a coordinate free form:

**Theorem 5.** Let $\phi$ be a $C^2$ function on a domain $D$, in $\mathbb{R}^3$. Suppose the level set $\phi = 0$ consists of two smooth surfaces $B_1$ and $B_2$ which intersect in a smooth curve $S$; suppose moreover that all other level sets of $\phi$ in $D$ are smooth. Let $P_0 \in S$, and let $L$ be a line through $P_0$, perpendicular to $S$, which bisects the tangent planes to the branches $B_1$ at $P_0$. Let $C$ be any solid cone in $D$ with vertex $P_0$ and directrix $L$, which does not touch the tangent planes to the branches except at $P_0$ itself. Then

$$\lim_{P \to P_0 \atop P \in C} \mu(\phi, P) = -\infty,$$

where $\mu(\phi, P)$ is the mean curvature at $P$ of the level surface of $\phi$ through $P$. Moreover this limit is approached uniformly on $C$.

---

\footnote{All that is required is that $\overline{C}$ be contained in the domain of definition of $\phi$, and that $|\nabla \phi(P)| \neq 0$ for all $P \in \overline{C}$ except $P = P_0$.}
let \( n(P) \) denote the value of the \( t \)s limit as \( P \to P_0 \) along this \( \gamma \). When the curve \( \gamma \) is on the unit vector in the line of \( C \) at some con-
the function \( n(P) \) restricted to \( C \) is con-
served by \( C \).

\[ w-axis; \text{for all these points } n(Q) \text{; exists an } \epsilon_1 > 0 \text{ such that if } \mu > \epsilon_0. \text{By the uniform continuity } \]

\[ |P-P'| < \epsilon_0 \text{ be sufficiently small so that } \]

\( \epsilon \) is the same \( w \)-coordinate as \( P \), we \( \epsilon_1 \).

\[ \exists \epsilon > \frac{|\mu|}{4} + \frac{|\mu|}{4} = \frac{|\mu|}{2}. \]

exists \( \epsilon_0 \) so that \( P \in C \), negative number).

\( \in C \) as \( P \to P_0 \). It follows \( P_0 \). We may summarize this in a coordinate free form:

\( \text{domain } D, \text{ in } \mathbb{R}^2. \) Suppose the \( \text{es } B_1 \) and \( B_2 \) which intersect in \( \phi \) \( P_0 \), perpendicular to \( S \), which \( t P_0 \). Let \( C \) be any solid cone in \( C \). Not touch the tangent planes to \( \infty. \)

the level surface of \( \phi \) through \( P \) \( \in C \).

\( \text{domain of definition of } \phi, \text{ and that } \)

\[ \text{Figure 13. An illustration of the argument arising in the study of the limit of the } \]

\( \text{mean curvatures, } \mu(\psi, P), \text{ as } P \to P_0 \text{ in } C. \)

\textbf{The main theorem.} In general, if \( \phi \) is a differentiable function whose level set \( \phi = 0 \) has singularities (for example as above, where \( \phi = 0 \) is singular along the curve \( S \)), the nearby level sets \( \phi = t (t \in \mathbb{R}, t \text{ small}) \) will be smooth. Thus we can think of the family of surfaces \( M_t: \phi = t \) as a canonical family of smooth approximations to \( M_0: \phi = 0 \) which converge to \( M_0 \) in some suitable geometric sense as \( t \to 0 \). However there are many ways of constructing such "smoothing families," and in fact generic perturbations of \( \phi \) will result in functions with smooth level surfaces near \( \phi = 0 \).

In addition to the \( \phi = t \) smoothing, we single out two other examples for specific mention at this point. We will return to them in more detail later (); the point here is to motivate the level of generality at which the Theorem 6 will be stated. First, there is "Gaussian smoothing," where for \( t > 0 \) we define

\[ f_t(x) = \frac{1}{\sqrt{2\pi t}} \int_D e^{-\|x-y\|^2/2t} \phi(y)dy. \]

The \( f_t(x) \) can be viewed as "smoothings of the function \( \phi \)" itself, and for small \( t, u \in \mathbb{R} \) the level sets \( f_t = u \) are smoothings of the surface \( \phi = 0 \). Note that \( \lim_{t \to 0} f_t = \phi \), and if we define \( f_0 = \phi \) the function \( f_t(x) \) depends differentiably on \( t \) and \( x \).
As a second example, start with a sequence of lattices (i.e. discrete subsets) \( L_i \) of \( D \), which get arbitrarily dense in \( D \) as \( i \to \infty \). We can use each lattice as a set of control points to construct a suitable bicubic spline \( f_i \), so that \( \lim_{i \to \infty} f_i = \phi \) and the level sets of the \( f_i \) near \( \phi = 0 \) are smooth. This example seems fundamentally different than that of the Gaussian smoothing; for one thing the "parameter" \( i \) is discrete, and there is no "natural" continuous parameter as in the Gaussian case. Note however that any "continuous parameter" smoothing \( f_i \) of \( \phi \) can be represented as a suitable sequence \( f_i \) converging to \( \phi \), without excessive information loss, at least at \( \phi \) itself. For this reason we will consider a sequence of functions \( f_i \) (with smooth level surfaces) which converge to \( \phi \) in \( C^2 \), as the most general practical formulation of a "smoothing family."

Theorem 6, which we prove in this section, states that when we smooth a transversal intersection in this sense, negative curvature at points of the smooth surfaces increases without bound the closer we get to the singular level set \( \phi = 0 \). Thus high negative curvature is seen to be the "stable form" of transversal intersections, in the sense that slight perturbations of the singularity yield smooth surfaces with arbitrarily high curvature. Indeed, it is impossible to actually detect an intersection in practice (if we have access to only one side of the surface); the most we can do is to measure curvature up to the order of \( 1/\sigma \), where \( \sigma \) is the limit of the resolving power of our measurement system.

**Theorem 6:** Let \( D \) be a domain in \( \mathbb{R}^3 \), and let \( \phi \in C^2(D) \) be a function whose level set \( \phi = 0 \) is the union of two smooth branches which intersect transversally in a smooth curve \( S \). Suppose \( \{ f_i \} \) is a sequence of functions which converge in \( C^2(D) \) to \( \phi \), and all the level sets of the \( f_i \) through any point \( P \in D - S \) are smooth. Let \( \gamma \) be a differentiable curve in \( D \) which intersects \( \phi = 0 \) at a single point \( P_0 \in S \), and which is not tangent to either of the branches of \( \phi = 0 \). Then

\[
\liminf_{\delta \to 0, i \to \infty} \{ \mu(f_i, P) \mid P \in \gamma, |P - P_0| \leq \delta \} = -\infty
\]

where \( \mu(f_i, P) \) denotes the mean curvature at \( P \) of \( M(f_i, P) \), i.e. the level set of \( f_i \) through \( P \).

**Remark:** If we let \( f_i = \phi \) for each \( i \) in Theorem 6, we get a form of Theorem 5.

**Proof of Theorem 6:** As in Proposition 1 of §3.1, for any \( P \in D - S \) we may write

\[
\mu(f_i, P) = \frac{-H(f_i, P) - N(f_i, P)H(f_i, P)N(f_i, P)}{|\nabla f_i(P)|}
\]

where \( H(f, P) \) denotes the Hessian normal to the level surface \( M(f, P) \), the direction of \( \nabla f(P) \). To sin \( \langle , \rangle_{t_i, P} \), and its trace by \( t_i(P) \). I \( \langle , \rangle \) and \( t(P) \) respectively. Moreover \( N_i(P) \), and \( N(\phi, P) \) by \( N(P) \).

Note that \( N(P_0) \) is not a \( \langle , \rangle \)

\[
\lim N(P) \text{ exists as } P \to P_0 \text{ along } \langle \phi = 0 \rangle \text{ and in particular the notation } N(P_0) \text{ to denote th}
\]

throughout the discussion there is

With our new notation, we have

\[
\mu(f_i, P) = \frac{t_i(P)}{\nabla f_i(P)}
\]

We add and subtract the numerator to obtain:

\[
\mu(f_i, P) =
\]

\[
(t(P_0) - \langle N(P_0), N(P_0) \rangle_{P_0}) + \langle \langle , \rangle_{P_0}, \langle , \rangle_{P_0} \rangle_{P_0}
\]

Now the term on the left in the numerator of the expression noting that our \( \gamma \) satisfies the 1 orientation of the axes. Therefor the term is a fixed negative number \( i \).

We now expand the term on tracting \( \langle N(P_0), N(P_0) \rangle_{P_0} \) to:

\[
\mu(f_i, P) = \frac{k}{|\nabla f_i(P)|} \]

\[
+ (t_i(P) - t(P_0))
\]

\[
- \langle N_i(P), N_i(P) \rangle \]

\[
+ \langle \langle N(P_0), N(P) \rangle \rangle
\]

In order to prove the theorem positive integer. Then there is
where $H(f, P)$ denotes the Hessian form of $f$ at $P$ and $N(f, P)$ is the unit normal to the level surface $M(f, P)$ at $P$, i.e. $N(f, P)$ is the unit vector in the direction of $\nabla f(P)$. To simply notation we will denote $H(f_i, P)$ by $(\cdot, \cdot)_i(P)$, and its trace by $t_i(P)$. $H(\phi, P)$ and $\text{tr}H(\phi, P)$ will be denoted by $(\cdot, \cdot)_P$ and $t(P)$ respectively. Moreover we will denote $N(f_i, P)$ simply by $N_i(P)$, and $N(\phi, P)$ by $N(P)$.

Note that $N(P_0)$ is not a priori defined. However by Proposition 3, $\lim N(P)$ exists as $P \to P_0$ along $\gamma$, since by hypothesis $\gamma$ is not tangent to the branches of $\phi = 0$ and in particular not tangent to $S$. We will therefore use the notation $N(P_0)$ to denote this limit for our given $\gamma$; since $\gamma$ is fixed throughout the discussion there is no ambiguity.

With our new notation, we have

$$
\mu(f_i, P) = \frac{t_i(P) - \langle N_i(P), N_i(P) \rangle_i(P)}{\| \nabla f_i(P) \|}
$$

We add and subtract the quantity $t(P_0) - \langle N(P_0), N(P_0) \rangle_{P_0}$ in the numerator to obtain:

$$
\mu(f_i, P) = \frac{(t_i(P) - \langle N_i(P), N_i(P) \rangle_i(P)) + (t_i(P) - t(P_0))}{\| \nabla f_i(P) \|}
$$

Now the term on the left in the numerator is the limit as $P \to P_0$ along $\gamma$ of the numerator of the expression for $\mu(\phi, 0)$ as in the proof of Proposition 4, noting that our $\gamma$ satisfies the hypotheses of that proposition for a suitable orientation of the axes. Therefore, by (*) in the proof of Proposition 4, this term is a fixed negative number $k$.

We now expand the term on the right in the numerator by adding and subtracting $\langle N(P_0), N(P_0) \rangle_{P_0}$ to obtain:

$$
\mu(f_i, P) = k/\| \nabla f_i(P) \| + \frac{(t_i(P) - t(P_0))/\| \nabla f_i(P) \|}{\| \nabla f_i(P) \|}
$$

In order to prove the theorem, we must show the following: let $L$ be any positive integer. Then there exist $n$, $\delta$ such that there is a $P \in \gamma$ with
\[ |P - P_0| \leq \delta \text{ and } \mu(f_i, P) < -L \text{ for all } i > n. \] In fact we will prove the stronger assertion

(6.2) Given \( L \), there exist \( n, \delta \) such that for \( i > n \) and all \( P \) satisfying

\[ \delta/2 \leq |P - P_0| \leq \delta, \mu(f_i, P) < -L. \]

We will denote

\[ \hat{1} = t_i(P) - t(P_0), \]
\[ \hat{2} = \langle N_i(P), N_i(P) \rangle_{i,P} - \langle N(P_0), N(P_0) \rangle_{i,P}, \]
\[ \hat{3} = \langle N(P_0), N(P_0) \rangle_{P_0} - \langle N(P_0), N(P_0) \rangle_{i,P}. \]

With this notation, (6.1) becomes

\[ \mu(f_i, P) = \frac{k + \hat{1} + \hat{2} + \hat{3}}{|\nabla f_i(P)|}. \]

Recalling that \( k \) is a fixed negative number (depending only on \( \gamma \)), to prove (6.2) we will find \( n, \delta \) such that for \( i > n \) and \( \delta/2 \leq |P - P_0| \leq \delta \), we have \( |\nabla f_i(P)| < |3k/2L|, |\hat{1}| < |k|/6, |\hat{2}| < |k|/6, |\hat{3}| < |k|/6. \) This will give the result, for then

\[ \mu(f_i, P) \leq \frac{k + |\hat{1}| + |\hat{2}| + |\hat{3}|}{|\nabla f_i(P)|} \]

\[ \leq \frac{k + |k|/6 + |k|/6 + |k|/6}{|\nabla f_i(P)|} \]

\[ = \frac{3k/2}{|\nabla f_i(P)|} \leq \frac{3k/2}{|3k/2L|} = -L. \]

Now, since \( f_i - \phi \in C^2 \), it follows that \( |\nabla f_i| - |\nabla \phi| \) uniformly on compact subsets of \( \gamma \), and similarly \( \langle \cdot, \cdot \rangle_{i,P} - \langle \cdot, \cdot \rangle \) uniformly on compact subsets of \( \gamma \) (for this purpose we may identify \( \langle \cdot, \cdot \rangle \) etc. with the appropriate Hessian matrix of second partial derivatives).

Let us first choose \( \delta_1 \) so that if \( |P - P_0| \leq \delta_1, |\nabla \phi(P)| < 3|k|/4L \); we can do this since \( |\nabla \phi(P_0)| = 0 \) and \( \nabla \phi \) is continuous. Next, choose \( n_1 \) so that, in view of the uniform convergence of \( \nabla f_i(P) \) to \( \nabla \phi(P) \) on the compact subset of \( \gamma \) defined by \( |P - P_0| \leq \delta_1, |\nabla f_i(P) - \nabla \phi(P)| \leq 3|k|/4L \) for \( P \) in this subset. It follows that for \( i > n_1 \) and \( |P - P_0| \leq \delta_1, |\nabla f_i(P)| < 3|k|/2L. \)

Now we look at the term \( \hat{1} \). We can write it \( (t_i(P) - t(P_0)) + (t(P) - t(P_0)) \). Recall that \( t_i(p) \) is the trace of \( \langle \cdot, \cdot \rangle_{i,P} \) and in particular it is a sum of second partial derivatives. Since the second derivatives of \( \phi \) are continuous, \( |t_i(P) - t(P_0)| < |k|/12 \) if \( P \) converge uniformly to \( P_0 \).

\[ |P - P_0| \leq \delta_2, \text{ we can find } P \] in this subset. It follows

We consider the term \( \hat{2} \) only on \( \gamma \) so is fixed by \( \gamma \). We can write \( \hat{2} = \langle B, B \rangle_{i,P} \). Choose \( \delta_3 \) (by the uniform convergence for which \( |P - P_0| \leq \delta_3 \)) so that \( |\hat{2}| < |k|/6. \) It follows that there

\[ \hat{3} = \langle N(P_0), N(P_0) \rangle_{P_0} - \langle N(P_0), N(P_0) \rangle_{i,P}. \]

Now, to study the \( \hat{2} = \langle N_i(P) + N(P_0), N_i(P) \rangle \) and let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator, let \( H \) denote the Hecke operator.

\[ \hat{2} = \langle N_i(P) + N(P_0), N_i(P) \rangle \]

(where the dot denotes \( \cdot \) and \( H \) is operating as a map)

\[ \hat{2} \leq |N_i(P) + N(P_0)| \leq 2 |H| \]

since \( N_i(P), N(P_0) \) are \( \cdot \)

It is a well known and it denotes the sum of the \( |H(v)| \leq |v|. \) Let \( i \) and \( H \) denote the Hessian matrices \( i = 1, 2, \ldots \); to see a bound for the Hessian \( H \) uniform convergence of \( |P - P_0| \leq \delta_2, \)

Now \( N(P) \) converge which is \( \leq \delta_3, \) and
derivatives of \( \phi \) are continuous, we can find a \( \delta_2 \) so that if \(|P - P_0| \leq \delta_2\), 
\(|t(P) - t(P_0)| < \frac{|k|}{12}\). Then, since the second partial derivatives of the 
\( f_i \) converge uniformly to those of \( \phi \) on the compact subset of \( \gamma \) defined by 
\(|P - P_0| \leq \delta_2\), we can find \( n_2 \) so that if \( i > n_2 \), 
\(|t_i(P) - t_i(P_0)| < \frac{|k|}{12}\) for 
P in this subset. It follows that for \( i > n_2 \) and \(|P - P_0| < \delta_2\), 
\( \Gamma < |k|/6\).

We consider the term \( \bar{3} \). Let \( \mathbf{B} \) denote the unit vector \( \mathbf{N}(P_0) \); it depends only on \( \gamma \) so is fixed throughout the discussion. Thus \( \bar{3} \) is 
\( \langle \mathbf{B}, \mathbf{B} \rangle \rho_0 - \langle \mathbf{B}, \mathbf{B} \rangle i, P \). We can write this 
\( \langle \mathbf{B}, \mathbf{B} \rangle \rho_0 - \langle \mathbf{B}, \mathbf{B} \rangle \rho_i + \langle \mathbf{B}, \mathbf{B} \rangle i, P \). Choose \( \delta_3 \) (by the continuity of the second partial derivatives of \( \phi \)) 
so that if \(|P - P_0| \leq \delta_3\), 
\( |\langle \mathbf{B}, \mathbf{B} \rangle \rho_0 - \langle \mathbf{B}, \mathbf{B} \rangle \rho_i| \leq |k|/12\). Then choose \( n_3 \) 
(by the uniform convergence of \( \langle \cdot, \cdot \rangle \perp P \) to \( \langle \cdot, \cdot \rangle \rho_i \) on the compact subset of \( \gamma \)) 
for which \(|P - P_0| \leq \delta_3\) so that if \( i > n_3 \), 
\( |\langle \mathbf{B}, \mathbf{B} \rangle \rho_i - \langle \mathbf{B}, \mathbf{B} \rangle i, P| < |k|/12\). It follows that there exist \( n_3, \delta_3 \) so that if \( i > n_3 \) and \(|P - P_0| \leq \delta_3\), 
\( \bar{3} < |k|/6\).

Now, to study the term \( \bar{2} \) we first write it in the form 
\( \bar{2} = (\mathbf{N}(i, P) + \mathbf{N}(P_0), \mathbf{N}(i, P) - \mathbf{N}(P_0)) \). Reverting back to matrix notation, 
let \( H \) denote the Hessian matrix of \( f_i \) at \( P \) in some coordinate system, 
and let \( N_i(P) \) and \( N(P_0) \) be written as vectors in the same system. Then 
\( \bar{2} = (\mathbf{N}(i, P) + \mathbf{N}(P_0), \mathbf{N}(i, P) - \mathbf{N}(P_0)) \).

(where the dot denotes ordinary dot product in the given coordinate system, 
and \( H \) is operating as a matrix on the vector \(\mathbf{N}(i, P) - \mathbf{N}(P_0)\)). Hence 
\( \bar{2} < |\mathbf{N}(i, P) + \mathbf{N}(P_0)| \cdot |H(\mathbf{N}(i, P) - \mathbf{N}(P_0))| \), 
\( \leq 2|H(\mathbf{N}(i, P) - \mathbf{N}(P_0))| \),

since \( \mathbf{N}(i, P), \mathbf{N}(P_0) \) are both unit vectors.

It is a well known and easily verified fact that if \( H \) is any matrix, and \( l \) 
denotes the sum of the lengths of the columns of \( H \), then for any vector \( \mathbf{v} \), 
\( |H(\mathbf{v})| \leq l|\mathbf{v}| \). Let \( \bar{l} \) denote an upper bound of the values of \( l \) 
for the Hessian matrices \( H = H(f_i, P) \) for \( P \in \gamma \), \(|P - P_0| \leq \delta_3\) and 
i = 1, 2, \cdots ; to see that this bound exists, observe that there is a similar bound 
for the Hessian of \( \phi \) at points \( P \) in this compact set, and then use the 
uniform convergence of the Hessians of the \( f_i \) to those of \( \phi \) on the set. Thus, 
if \(|P - P_0| \leq \delta_3\), 
\( |\bar{2}| \leq 2l|\mathbf{N}(i, P) - \mathbf{N}(P_0)| \),
\( \leq 2l(|\mathbf{N}(i, P) - \mathbf{N}(P)| + |\mathbf{N}(P) - \mathbf{N}(P_0)|) \).

Now \( \mathbf{N}(P) \) converges to \( \mathbf{N}(P_0) \) as \( P \to P_0 \) along \( \gamma \). Therefore, choose a \( \delta_4 \) 
which is \( < \delta_3 \), and which also has the property that if
\[ |P - P_0| \leq \delta_4 |N(P) - N(P_0)| \leq |k|/(24\bar{I}) \]. Observe that away from \( S \) (i.e. away from \( P_0 \) in our situation we are just working on \( \gamma \), \( N(P) \) is a well-defined vector-valued function of \( P \), expressible in terms of the derivatives of \( \phi \). Thus on any compact set not meeting \( S \), \( N_i(P) \) converges uniformly to \( N(P) \). In particular, there exists \( n_4 \) so that if \( i > n_4 \) and \( \delta_4/2 \leq |P - P_0| \leq \delta_4 \), then \( |N_i(P) - N(P)| < |k|/(24\bar{I}) \). It follows that there exist \( n_4 \) and \( \delta_4 \) so that if \( i > n_4 \) and \( \delta_4/2 \leq |P - P_0| \leq \delta_4 \), \( |2| < |k|/6 \).

It is now clear that if we let \( \delta = \inf(\delta_1, \delta_2, \delta_3, \delta_4) \) and \( n = \sup(n_1, n_2, n_3, n_4) \) the conclusion of 6.2 is valid. This completes the proof of Theorem 6.

**Remark.** Suppose that we have a smoothing family \( f_i \to \phi \) as in Theorem 6. We would like to conclude that for sufficiently large \( i \), the level surfaces of the \( f_i \) close to \( \phi = 0 \) contain contours which are part boundaries in the sense of the Negative Minima Partitioning Rule of \( \S 2 \) above, and moreover that these contours converge to \( S \) in some reasonable way. Theorem 6 makes this very plausible, but does not go all the way to give a proof. The essential difficulties here may be understood by considering the case of the level surface smoothing of \( \phi \), i.e., the case where all the \( f_i \) are just \( \phi \) itself. As we have seen, these level surfaces have points of arbitrarily high negative curvature close to \( S \). The problem lies in the possibility that the relevant lines of curvature on the given level surfaces near \( S \) get “trapped” in neighborhoods of \( S \). In this way the line of curvature might approach \( S \) so that the points on it could have ever increasing negative curvature, i.e., no points is an extremum. Since the boundary contours by definition consist of points which are extrema of curvature on their corresponding lines of curvature, there would be no such contour in this case. Happily this kind of pathology can be ruled out; the methods are beyond the scope of this paper and will be published elsewhere.

4. **EXAMPLES OF PARTITIONS**

In the previous section we proved that smoothing a transversal intersection leads to large curvature (negative for solid union, positive for solid subtraction) regardless of how one smooths. In this section we determine analytically the negative minima partitioning contours on several classes of surfaces. This allows a more rigorous understanding of the rule and the boundaries it defines. In particular, this section illustrates that the negative minima rule is a 3-D definition of part boundary, not a 2-D rule of thumb for finding 2-D parts (such as the “matched concavities heuristic,”—see Brady & Asada, 1984, for a description, and critique, of the matched concavities heuristic).

**Decomposition of developable**

Developable surfaces are generated by a one parameter family of lines \( \{ \text{the parameter } u^1 \in (a, b) \subseteq v(u^1) \neq 0, \text{ such that both } \} \) each \( u^1 \in (a, b) \), the line \( L(v(u^1)) \) is called the line of the parameter \( f \) surface is given by the parametrization:

\[ x(u^1, u^2) = p(u^1) \]

The curve \( p(u^1) \) is called the surface \( x \). In what follows is the arc length along \( p \) and the following notation: If \( w \) is a \( u^1, u^2 \), then \( w \) will denote \( x_2 = \partial x/\partial u^2 \), etc. A ruled surface \( (v, v_1, p_1) = 0 \) has \( p_1 \) all lie in a single plane.

In the next two subsections by the minima rule for two cylinder and cone.

**Cylinders**

A cylinder is a developable plane and whose rulings, \( v(1) \), \( v(2) \), \( v(1) = 0 \). For \( t \)

\[ x_1 = p_1 + u^2 v_1 = p_1 \]

The surface normal is

\[ N = \frac{x_1}{|x_1|} \]

The second fundamental
Decomposition of developable surfaces

Developable surfaces are a special case of ruled surfaces, surfaces that are generated by a one parameter family of lines (Do Carmo, 1976). A one parameter family of lines \{p(u^1), v(u^1)\} is a correspondence that assigns to the parameter \( u^1 \in (a, b) \subset \mathbb{R} \) a point \( p(u^1) \in \mathbb{R}^2 \) and a vector \( v(u^1) \in \mathbb{R}^3 \), \( v(u^1) \neq 0 \), such that both \( p(u^1) \) and \( v(u^1) \) depend differentiably on \( u^1 \). For each \( u^1 \in (a, b) \), the line \( L(u^1) \) which passes through \( p(u^1) \) and is parallel to \( v(u^1) \) is called the line of the family at \( u^1 \).

Given a one parameter family of lines \{p(u^1), v(u^1)\}, the associated ruled surface is given by the parameterization

\[
x(u^1, u^2) = p(u^1) + u^2 v(u^1), \quad u^1 \in (a, b) \subset \mathbb{R}, \quad u^2 \in \mathbb{R}.
\]

The curve \( p(u^1) \) is called a directrix, and the lines are called the rulings of the surface \( x \). In what follows we assume, without loss of generality, that \( u^1 \) is arc length along \( p \) and that \( |v(u^1)| = 1 \). Moreover we will adopt the following notation: if \( w \) is a vector or scalar valued function of the parameters \( u^1, u^2 \), then \( w_i \) will denote \( \partial w/\partial u^i \). Thus \( v = \partial v/\partial u^1, p_{11} = \partial^2 p/\partial u^1 \partial u^2, x_2 = \partial x/\partial u^2 \) etc. A ruled surface is said to be developable if the scalar triple product \( (v, v_1, p_1) = 0 \) everywhere on the surface, implying that \( v, v_1 \), and \( p_1 \) all lie in a single plane.

In the next two subsections we determine the partitioning contours defined by the minima rule for two nonexhaustive cases of developable surfaces—the cylinder and cone.

Cylinders

A cylinder is a developable surface whose directrix, \( p \), lies entirely in one plane and whose rulings, \( v(u^1) \), are parallel to a fixed direction in \( \mathbb{R}^3 \), implying that \( v_1 = 0 \). For a cylinder the first partial derivatives are \( x_1 = p_1 + u^2 v_1 = p_1 \) (since \( v_1 = 0 \)) and \( x_2 = v \). The metric tensor is then

\[
g_{ij} = x_i \cdot x_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

The surface normal is

\[
N = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{p_1 \times v}{|p_1 \times v|} = p_1 \times v
\]

The second fundamental coefficients are
\[ b_{ij} = x_{ij} \cdot N = \left[ \begin{array}{cc} (p_{11}, p_{1}, v) & 0 \\ 0 & 0 \end{array} \right] + \left[ \begin{array}{c} |p_{11}| \\ 0 \end{array} \right] \]

Since \( g_{12} = b_{12} = 0 \) the principal curvatures on a cylinder are

\[ \kappa_1 = b_{11}/g_{11} = |p_{11}| \]
\[ \kappa_2 = b_{22}/g_{22} = 0 \]

The expression for \( \kappa_1 \) is the magnitude of the second derivative of \( p \) with respect to arc length (with sign determined by the orientation of the field of surface normals) which is simply the curvature along the directrix \( p \). The directrix and its translations are, in fact, one set of lines of curvature and the rulings the other set, since \( g_{12} \) and \( b_{12} \) are zero. As expected, the curvature along the rulings, \( \kappa_2 \), is zero. Consequently no partitioning contours arise from the rulings (since there are no extrema of the principal curvature \( \kappa_2 \)). Only the minima of \( \kappa_1 \) along the directrix and its translations are used for defining part boundaries.

Figure 2, as discussed in the introduction, shows a cylinder and its partitioning contours (dotted lines) for one of the orientations of the field of surface normals. The partitioning contours break the cylinder into parts that seem natural enough. If one inverts the figure one will experience a figure-ground reversal, causing the bumps of the surface to become dips and vice-versa. Notice that when the figure and ground reverse the perceived partitioning lines shift away from the indicated dotted lines and to the lines that were previously positive maxima of \( \kappa_1 \). This occurs because the figure-ground reversal is associated with a reversal in the orientation of the field of surface normals and, hence, in the sign of \( \kappa_1 \) everywhere on the surface. Contours of positive maxima of \( \kappa_1 \) and contours or negative minima of \( \kappa_1 \) swap places and the partitioning along the new negative minima becomes apparent.

As noted in the introduction, segmentation rules which use only the Gaussian curvature, rather than analyzing the principal curvatures independently, fail on this example and on cones because the Gaussian curvature is everywhere zero, making impossible any segmentation based only upon the Gaussian curvature. Yet human observers readily and consistently perceive partitions in surfaces whose Gaussian curvature is everywhere zero.

Cones

Cones are a special case of ruled surfaces in which the directrix, \( p \), is simply a point, the vertex of the cone. In consequence one can give the following parametrization for the cone:

\[ x(u^1, u^2) = u^2 v(u^1), \quad u^1 \in (a, b) \subset \mathbb{R}, \quad u^2 \in \mathbb{R}. \]

Figure 14. Partitions of a
For this parametrization the first partial derivatives are \( x_1 = u^2 v_1 \) and \( x_2 = v \). The metric tensor is

\[
g_{ij} = x_i \cdot x_j = \begin{pmatrix}
(u^2)^2 (v \cdot v) & 0 \\
0 & 1
\end{pmatrix}
\]

The surface normal is

\[
N = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{u^2 (v_1 \times v)}{u^2 |v_1 \times v|} = \frac{v_1 \times v}{|v_1|}
\]

The second fundamental coefficients are

\[
b_{ij} = x_{ij} \cdot N = \begin{pmatrix}
(u^2 (v_1, v, v_{11})/|v_1| & 0 \\
0 & 0
\end{pmatrix}
\]

Since \( g_{12} = b_{12} = 0 \) the principal curvatures on a cone are

\[
\kappa_1 = \frac{b_{11} / g_{11}}{(u^2)^2 (v_1 \cdot v_1) |v_1|} = \frac{(v_1, v, v_{11})}{u^2 |v_1|^3}
\]

\[
\kappa_2 = \frac{b_{22} / g_{22}}{0} = 0
\]

The \( u^1 \)- and \( u^2 \)-parameter curves are lines of curvature, since \( g_{12} \) and \( b_{12} \) are both zero. As one would expect, the principal curvature, \( \kappa_2 \), along the \( u^2 \)-parameter curves is everywhere zero. The expression for \( \kappa_1 \) along the \( u^1 \)-parameter curves (where \( u^2 \) is constant) does not depend on \( u^2 \). Thus the contours of negative minima of \( \kappa_1 \) are straight lines which radiate from the vertex of the cone. An example cone is shown in Figure 14 with the partitioning contours indicated by dotted lines. The resulting parts appear quite natural.

\[\text{Figure 14.} \quad \text{Partitions of a cone}\]
Surfaces of revolution

A surface of revolution is a set $S \subset \mathbb{R}^3$ obtained by rotating a regular plane curve, $\alpha$, about an axis in the plane which does not meet the curve. Let the $xz$-plane be the plane of $\alpha$ and let the axis of rotation be the $z$-axis. Let

$$\alpha(u^1) = (x(u^1), z(u^1)), \quad a < u^1 < b, \quad x(u^1) > 0,$$

and let $u^2$ be the rotation angle about the $z$-axis. Then we obtain a map

$$x(u^1, u^2) = (x(u^1) \cos(u^2), x(u^1) \sin(u^2), z(u^1))$$

from the open set $U = \{(u^1, u^2) \in \mathbb{R}^2; \ 0 < u^2 < 2\pi, \ a < u^1 < b\}$ into $S$ (as shown in Figure 15). The curve $\alpha$ is called the generating curve of $S$, and the $z$-axis is the rotation axis of $S$. The circles swept out by the points of $\alpha$ are called the parallels of $S$, and the various placements of $\alpha$ on $S$ are called the meridians of $S$.

The partial derivatives are $x_1 = (x_1 \cos(u^2), x_1 \sin(u^2), z_1)$ and $x_2 = (-x \sin(u^2), x \cos(u^2), 0)$. The metric tensor is

$$g_{ij} = x_i \cdot x_j = \begin{bmatrix} x_1^2 + z_1^2 & 0 \\ 0 & x^2 \end{bmatrix}.$$ 

The surface normal is

$$N = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{(z_1 \cos(u^2), z_1 \sin(u^2), -x_1)}{\sqrt{z_1^2 + x_1^2}}.$$

If we let $u^1$ be arc length along $\alpha$ then $\sqrt{z_1^2 + x_1^2} = 1 = g_{11}$ and

$$N = (z_1 \cos(u^2), z_1 \sin(u^2), -x_1).$$

Figure 15. A convenient parametrization for a surface of revolution.

The second fundamental form

$$b_{ij} = x_{ij} \cdot \nu$$

Since $g_{12} = b_{12} = 0$ the properties of $\kappa_1$ are:

$$\kappa_1 = \kappa.$$  

The expression for $\kappa_1$ is

$$\kappa_1 = \frac{1}{\rho},$$

where $\rho$ is the radius of curvature. In particular, if $\rho$ is constant, then the surface is a sphere. Consequently, no part of the surface appears to be curved along the meridians.

Figure 16. Partitions of some surfaces.
The second fundamental coefficients are
\[ b_{ij} = x_{ij} \cdot \mathbf{N} = \begin{pmatrix} x_{11} z_1 - s_1 z_{11} & 0 \\ 0 & -x z_1 \end{pmatrix}. \]
Since \( g_{12} = b_{12} = 0 \) the principal curvatures on a surface of revolution are
\[ \kappa_1 = b_{11} / g_{11} = x_{11} z_1 - x_1 z_{11}, \]
\[ \kappa_2 = b_{22} / g_{22} = -z_1 / \kappa. \]

The expression for \( \kappa_1 \) is identical to the expression for the curvature along \( \alpha \). In fact the meridians (the various positions of \( \alpha \) on \( S \)) are lines of curvature, as are the parallels. The curvature along the meridians is given by the expression for \( \kappa_1 \) and the curvature along the parallels is given by the expression for \( \kappa_2 \). The expression for \( \kappa_2 \) is simply the curvature of a circle of radius \( x \) multiplied by the cosine of the angle that the tangent to \( \alpha \) makes with the axis of rotation.

Observe that the expressions for \( \kappa_1 \) and \( \kappa_2 \) depend only upon the parameter \( u^1 \), not \( u^2 \). In particular, since \( \kappa_2 \) is independent of \( u^2 \) there are no extrema or inflections of the normal curvature along the parallels. The parallels are circles. Consequently no partitioning contours arise from the lines of curvature associated with \( \kappa_2 \). Only the minima of \( \kappa_1 \) along the meridians are used for partitioning. Figure 16 shows several surfaces of revolution with the minima of curvature along the meridians marked. The resulting partitioning contours appear natural.

Figure 1, as discussed in the introduction, illustrates that reversing the orientation (of the field of surface normals) of a surface of revolution causes us to carve the same surface differently. The dotted circular lines in the figure

![Figure 16. Partitions of some surface of revolution](image)
are the partitioning contours according to the negative minima rule. Note that they lie in the valleys of the top figure. If the figure is inverted they no longer lie in the valleys but on the peaks. By reversing the field of surface normals the signs of the principal curvatures everywhere have reversed. Contours of negative minima of the principal curvatures become contours of positive maxima, and vice-versa. Consequently the part boundaries are not invariant under a reversal of orientation.

The torus
A torus is a surface in $\mathbb{R}^3$ which is obtained by revolving a circle about a line not passing through the circle, as shown in Figure 17. A convenient parameterization for the torus is

$$x(u^1, u^2) = ((b + a \sin(u^2)) \cos(u^1), (b + a \sin(u^2)) \sin(u^1), a \cos(u^2)),$$

$b > a$.

The first partials are $x_1 = (-(b + a \sin(u^2)) \cos(u^1), (b + a \sin(u^2)) \cos(u^1), 0)$ and $x_2 = (a \cos(u^2) \cos(u^1), a \cos(u^2) \sin(u^1), -a \sin(u^2))$. The metric tensor is

$$g_{ij} = x_i \cdot x_j = \begin{pmatrix} (b + a \sin(u^2))^2 & 0 \\ 0 & a^2 \end{pmatrix}.$$ 

The surface normal is

$$N = \frac{x_1 \times x_2}{|x_1 \times x_2|} = (-\cos(u^1) \sin(u^2), -\sin(u^1) \sin(u^2), -\cos(u^2)).$$

The second fundamental coefficients are

$$b_{ij} = x_{ij} \cdot N = \begin{pmatrix} (b + a \sin(u^2)) \sin(u^2) & 0 \\ 0 & a \end{pmatrix}.$$ 

Since $g_{12} = b_{12} = 0$ the $u^1$- and $u^2$-parameter curves are lines of curvature and the principal curvatures on a torus are

$$\kappa_1 = b_{11}/g_{11} = \sin(u^2)/(b + a \sin(u^2)),
\kappa_2 = b_{22}/g_{22} = a^{-1}.$$ 

The principal curvature $\kappa_1$ is associated with the $u^1$-parameter curves and $\kappa_2$ with the $u^2$-parameter curves. $\kappa_2$ is a constant, so the torus is not partitioned using the $u^2$-parameter independent of $u^1$. The parameter lines of curve indivisible unit based on the flattening it slightly along on that the circular part elliptical and bowed slight to test this against percept. Figure 18 illustrates a flattened surface of $f(u^1)$.

Let $f(u^1)$ be abbreviated with respect to $u^1$. Then

$$g_{ij} = x_i \cdot x_j = \begin{pmatrix} (f')^2(\cos^2(u^2) + f' \sin(u^2) \cos(u^2)) & f' \sin(u^2) \cos(u^2) \\ f' \sin(u^2) \cos(u^2) & f' \sin^2(u^2) \end{pmatrix}.$$ 

Figure 17. The torus has $n$
negative minima rule. Note that in the figure is inverted they no longer use the field of surface normals there have reversed. Contours of positive max-

boundaries are not invariant under

by revolving a circle about a line figure 17. A convenient parametr-

\[ a \sin(u^2) \sin(u^1), a \cos(u^2) \],

\[ b + a \sin(u^2) \]

\[ a \cos(u^2) \sin(u^1), -a \sin(u^2) \].

\[ \sin(u^2) = 0 \]

\[ a^2 \]

\[ -\sin(u^1) \sin(u^2), -\cos(u^2) \].

\[ \begin{pmatrix} a^2 & 0 \\ 0 & a \end{pmatrix} \]

ameter curves are lines of curvature

\[ b + a \sin(u^2) \],

\[ a^{-1} \].

with the \( u^1 \)-parameter curves and

constant, so the torus is not parti-

tioned using the \( u^2 \)-parameter lines of curvature. \( \kappa \) is not a constant, but it is independent of \( u^1 \). Therefore the torus is not partitioned using the \( u^1 \)-parameter lines of curvature either. We conclude that the torus is one indivisible unit based on the negative minima partitioning rule.

Flattened surfaces of revolution

What happens to the partitioning contours on a surface of revolution if we flatten it slightly along one axis orthogonal to the axis of revolution? We show here that the circular partitioning contours of the surface of revolution become elliptical and bowed slightly up or down in the middle. It would be of interest to test this against perceptual judgments.

Figure 18 illustrates a convenient parametrization for a surface of revolution which is flattened:

\[ x(u^1, u^2) = (f(u^1) \cos(u^2), af(u^1) \sin(u^2), (u^1)), \quad 0 < a < 1. \]

Let \( f(u^1) \) be abbreviated to \( f \) and let primes over the \( f \)'s indicate derivative with respect to \( u^1 \). Then the metric tensor is

\[ g_{ij} = x_i \cdot x_j \]

\[ = \begin{pmatrix} (f^2(\cos^2(u^2) + a^2 \sin^2(u^2))) + 1 & f f' \sin(u^2) \cos(u^2)(a^2 - 1) \\ f f' \sin(u^2) \cos(u^2)(a^2 - 1) & f^2(\sin^2(u^2) + a^2 \cos^2(u^2)) \end{pmatrix} \]
Figure 18. Partitions on a flattened surface of revolution

The second fundamental coefficients are

$$b_{ij} = x_{ij} \cdot N = \begin{bmatrix} -af/d & 0 \\ 0 & af/d \end{bmatrix}$$

where $$d = \sqrt{a^2 \cos^2(u^2) + \sin^2(u^2)} = a^2(f)^2$$.

Since $$x_1 \cdot x_2 \neq 0$$ in general, the parameter curves are not in general lines of curvature. However when $$f = 0$$ then $$x_1 \cdot x_2 = 0$$ so that contours where this holds are lines of curvature. These contours are elliptical cross sections of the flattened surface of revolution, cross sections having either the greatest or least major axis locally. Along these lines of curvature the associated principal curvature is

$$\kappa = \frac{b_{22}}{b_{22}} = af^{-1}(\sin^2(u^2) + a^2 \cos(u^2))^{-3/2}.$$ 

Its extrema occur when

$$\frac{\partial \kappa}{\partial u^2} = -7/2af^{-1}(a^2 \cos^2(u^2) + \sin^2(u^2))^{-5/2}(2 \cos(u^2) \sin(u^2)$$

$$- 2a^2 \cos(u^2) \sin(u^2)) = 0,$$

which happens when $$a^2 \cos(u^2) \sin(u^2) = \cos(u^2) \sin(u^2)$$. This implies that $$u^2 = n\pi/2$$, for $$n$$ an integer. For $$n$$ even, $$\kappa = a^{-2}f^{-1}$$, and for $$n$$ odd, $$\kappa = af^{-1}$$. Thus the minima occur when $$u^2 = \pi/2$$ or $$3\pi/2$$. However these minima are positive minima, since $$a$$ and $$f$$ are both positive, and consequently there are no partitioning contours which arise from this family of lines of curvature.

To determine the pair of curvature, we begin becomes

for $$n$$ even, and

for $$n$$ odd. Thus the $$u^1$$ curvature. These curves are lution with the $$xz$$-plane curvature is

$$\kappa =$$

and for $$n$$ odd it is

$$\kappa =$$

The extrema of these values of $$u^1$$ (because a surface of revolution are is flattened, the partition elliptical and usually bo

Elbows

An apparent problem "elbow", the problem b of curvature are not clo for smooth elbows and, ;

As can be seen in Fig specify an incomplete p the partition is inherentl in the figure is equally rc

Elongated torus

Elbows may also occur which has been scaled a
To determine the partitioning contours defined by the other family of lines of curvature, we begin by noting that when \( u^2 \) is \( n\pi/2 \) the metric tensor becomes

\[
  g_{ij} = \begin{pmatrix}
  (f')^2 + 1 & 0 \\
  0 & a^2 f^2 
  \end{pmatrix}
\]

for \( n \) even, and

\[
  g_{ij} = \begin{pmatrix}
  a^2(f')^2 + 1 & 0 \\
  0 & f^2 
  \end{pmatrix}
\]

for \( n \) odd. Thus the \( u^1 \)-parameter curves given by \( u^2 = n\pi/2 \) are lines of curvature. These curves are also the intersection of the flattened surface of revolution with the \( xz \)-plane or \( yz \)-plane. For \( n \) even the associated principal curvature is

\[
  \kappa = b_{11}/g_{11} = -f''(1 + (f')^2)^{-3/2},
\]

and for \( n \) odd it is

\[
  \kappa = b_{11}/g_{11} = -af''(1 + a(f')^2)^{-3/2}.
\]

The extrema of these two curvatures do not, in general, occur at the same values of \( u^1 \) (because \( a \neq 1 \)). Thus the partitioning contours on the flattened surface of revolution are not, in general, planar. So as a surface of revolution is flattened, the partitioning contours which are at first circles become more elliptical and usually bow either up or down slightly.

**Elbows**

An apparent problem for the negative minima partitioning rule is the "elbow", the problem being that for elbows the contours of negative minima of curvature are not closed, so no parts are uniquely delimited. This is true for smooth elbows and, as shown in Figure 19, for nonsmooth elbows as well.

As can be seen in Figure 19, however, there is good reason for the rule to specify an incomplete partitioning contour—the appropriate way to continue the partition is inherently ambiguous. Each of these three completions shown in the figure is equally reasonable.

**Elongated torus**

Elbows may also occur on entirely smooth surfaces. For instance, the torus which has been scaled along one axis has two elbows. The following deriva-
Shape Decompositions: The Role of...

tion will show that the nega-
lar contours, one on the insi-
The elongated torus may
\[ x(u^1, u^2) = ((b + a \sin(u^2)) \cos(u) \]

This corresponds in Figure first partials are
\[ x_1 = (-b + a \sin(u^2)) \cos(u) \]
and
\[ x_2 = (a \cos(u^2)) \sin(u) \]
The metric tensor is
\[
\begin{bmatrix}
(b + a \sin(u^2))^2 & (a \cos(u^2) \cos(u^1) \sin(u^1) + (b + a) \sin(u^2) \cos(u^1) \sin(u^1)) \\
(a \cos(u^2) \cos(u^1) \sin(u^1)) & a \cos(u^2) \cos(u^1) \sin(u^1)(b + a)
\end{bmatrix}
\]
The surface normal is
\[ N = (-d \sin(u^2)) \cos(u) \]
where
\[ f = \sqrt{d^2 \sin^2(u^2) \cos^2(u)} \]

Figure 19. Partitioning of an elbow

Figure 20. An elongated torus
tion will show that the negative minima rule gives rise to two open semicircular contours, one on the inside of each elbow, as shown in Figure 20.

The elongated torus may be conveniently parametrized as

\[ x(u^1, u^2) = ((b + a \sin(u^2)) \cos(u^1), d(b + a \sin(u^2)) \sin(u^1), a \cos(u^2)), \quad b > a, d > 1. \]

This corresponds in Figure 17 to expanding the torus along the \( x^2 \)-axis. The first partials are

\[ x_1 = (-b + a \sin(u^2)) \sin(u^1), d(b + a \sin(u^2)) \cos(u^1), 0) \]

and

\[ x_2 = (a \cos(u^2) \cos(u^1), ad \cos(u^2) \sin(u^1), -a \sin(u^2)). \]

The metric tensor is

\[
\begin{pmatrix}
(b + a \sin(u^2))^2 \sin^2(u^1 + d^\theta \cos^2(u^1)) & a \cos^2(u^2) \cos^2(u^1) \sin(u^1) (b + a \sin(u^2))(d^2 - 1) \\
(a \cos(u^2) \cos(u^1) \sin(u^1) (b + a \sin(u^2))(d^2 - 1) & a^2 \cos^2(u^2) \sin^2(u^1) + d^2 \cos^2(u^2) \sin^2(u^1) + \sin^2(u^2)
\end{pmatrix}
\]

The surface normal is

\[ N = (-d \sin(u^2) \cos(u^1), -\sin(u^1) \sin(u^2), -d \cos(u^2))/f, \]

where

\[ f = \sqrt{d^2 \sin^2(u^2) \cos^2(u^1) + \sin^2(u^1) \sin^2(u^2) + d^2 \cos^2(u^2)}. \]

Figure 20. An elongated torus has two semi-circular contours of partition.
The second fundamental coefficients are

\[
\mathbf{b}_{ij} = \begin{bmatrix} \frac{d \sin(u^2)}{f} & 0 \\ 0 & \frac{d \sin(u^2)}{f} \end{bmatrix}.
\]

Since \( g_{12} \neq 0 \) the \( u^1 \)- and \( u^2 \)-parameter curves are not in general lines of curvature. However, along the curve \( u^2 = \pi/2 \) we have \( \cos(u^2) = 0 \), \( \sin(u^1) = 1 \), and

\[
\mathbf{g}_{ij} = \begin{bmatrix} (b + a)^2 (\sin^2(u^1) + d^2 \cos^2(u^1)) & 0 \\ 0 & a^2 \end{bmatrix},
\]

implying that this is a line of curvature (\( b_{12} \) and \( g_{11} \) are zero). The second fundamental coefficients are

\[
\mathbf{b}_{ij} = \begin{bmatrix} (b + a) & 0 \\ 0 & \frac{1}{a} \end{bmatrix},
\]

where

\[
h = \sqrt{d^2 \cos^2(u^1) + \sin^2(u^1)}.
\]

The principal curvature along this line of curvature is

\[
\kappa = \frac{b_{11}}{g_{11}} = \frac{d(b + a)^{-1}}{a^{-1}} h^{-1}.
\]

The extrema of curvature along this line occur where \( \partial \kappa / \partial u^1 = 0 \).

\[
\frac{\partial \kappa}{\partial u^1} = -3/2 d(b + a)^{-1} (d^2 \cos^2(u^1)) + \sin^2(u^1)^{-1/2} (2 \sin(u^1) \cos(u^1) - 2 d^2 \sin(u^1) \cos(u^1)) = 0.
\]

Since \( d > 1 \) this implies that \( d^2 \cos(u^1) \sin(u^1) = \cos(u^1) \sin(u^1) \), which occurs for \( u^1 = n \pi/2 \), where \( n \) is an integer. Positive maxima of curvature occur when \( n \) is odd, negative minima when \( n \) is even.

A similar analysis shows that the contour \( u^2 = -\pi/2 \) is a line of curvature whose extrema occur for \( u^1 = n \pi/2 \), where \( n \) is an integer. The difference is that the positive maxima of curvature occur when \( n \) is even, negative minima when \( n \) is odd.

Finally, at the parameter point \( (\pi/2, 0) \) we have that \( \sin(u^1) = 1 \), \( \cos(u^1) = 0 \), \( \sin(u^2) = 0 \), \( \cos(u^2) = 1 \), and find that the metric tensor is

implying that at this point directions. The second fun

Hence \( \kappa_1 = b_{11}/g_{11} = 0 \) ter points \((-\pi/2, 0)\), \(-\pi/2, 0)\), then, we have found the minimum of \( \kappa_1 \), the out lowermost points have \( \kappa_1 \) tioning contours at the \( \pi < u^2 < 0 \), and \( u^1 = \)

5. SUMMARY

To recognize an object from into parts. Defining part afford the broadest possible generic, stable property boundary-based partition that when one smooths a continually large curvature a solid union, positive for \( i \) propose, in consequence, pal curvatures and some tures are used by the \( h \) which tell when to use minima are being developed theory to multiple scales

REFERENCES

\[
\begin{bmatrix}
0 & \frac{d}{d\ell} f \\
\frac{d}{d\ell} f & 0
\end{bmatrix},
\]
es are not in general lines of \(\pi/2\) we have \(\cos(u^2) = 0\),
\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]
\(g_{11}\) are zero). The second
\[
\begin{bmatrix}
0 & \frac{d}{d\ell} f \\
\frac{d}{d\ell} f & 0
\end{bmatrix},
\]
\(\vec{r}'(u^1)\).
\[\frac{d}{d\ell} f \frac{d}{d\ell} f = -1, h^{-3}.
\]
\(\lambda_{(1)}\) where \(\lambda_{(1)} = 0\).
\[
\begin{bmatrix}
u^1 - 2d^2 \sin(u^1) \cos(u^1)) = 0.
\end{bmatrix}
\]
\[\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
\(b_{ij} = \begin{bmatrix}
0 & 0 \\
0 & ad
\end{bmatrix}
\]
\(g_{ij} = \begin{bmatrix}
b^2 & 0 \\
0 & a^2d^2
\end{bmatrix}
\)
implies that at this point the \(u^1\) and \(u^2\)-parameter curves are in principal
directions. The second fundamental form is
\[\kappa = b_{11}/g_{11} = 0.\]
By symmetry this also holds for \(\kappa\) at the parameter points \((-\pi/2, 0), (-\pi/2, \pi), (\pi/2, \pi)\). At each of the two elbows, then, we have found that the innermost point of the elbow is a negative
minimum of \(\kappa\), the outermost is a positive maximum, the uppermost and lowermost points have \(\kappa = 0\). By symmetry we conclude that the two partitioning contours at the elbows are the open semicircles \(u^1 = \pi/2, \pi < u^2 < 0\), and \(u^1 = -\pi/2, \pi < u^2 < 0\).

5. SUMMARY

To recognize an object from its shape it is useful first to decompose the shape
into parts. Defining parts by their boundaries, rather than by their shapes,
afford the broadest possible scope to the partitioning scheme. Transversality, a
generic, stable property of the intersection of surfaces, motivates the boundary-based
partitioning scheme considered here. In particular we show that when one smooths a transversal intersection of surfaces one obtains arbitrarily large curvature as the intersection curve is approached (negative for solid union, positive for solid subtraction) regardless of how one smooths. We propose, in consequence, that some contours of negative minima of the principal curvatures and some contours of positive maxima of the principal curvatures are used by the human visual system as part boundaries. The rules which tell when to use positive maxima and when instead to use negative minima are being developed. Also to be developed is an extension of this theory to multiple scales of resolution.

REFERENCES


**1. INTRODUCTION**

The problem of recognizing “internal representation” of an object from a single view can be considered in the sense that a single view can possibly convey all the information about the object with sufficient a priori knowledge. An example may illustrate this idea: take a few views of the object when viewed from different viewpoints. When you take a few step views, you can see qualitative differences between the views separately. I call the qualitative differences between the views the “parts” of an object. This work was supported by...